Resonant Alfvén Wave Excitation in Two-Dimensional Systems: 
Singularities in Partial Differential Equations

MICHAEL J. THOMPSON
Astronomy Unit, School of Mathematical Sciences, Queen Mary and Westfield College, London, England

ANDREW N. WRIGHT
Department of Mathematical and Computational Sciences, University of St. Andrews, Fife, Scotland

The resonant excitation of Alfvén waves is considered in a cold plasma embedded in a uniform magnetic field $B_0$. All wave fields are assumed to vary as $\exp[I(\omega y - \omega t)]$, and the background medium is invariant in $y$. The background density distribution $\rho_0(x,z)$ is otherwise completely arbitrary. Regular and singular solutions for the waves are derived systematically in the vicinity of a resonance by considering a generalized Frobenius series, and we are able to recover many results found in earlier studies. Some new features of our work include a generalization of the overlap integral determining the efficiency with which any resonance may be excited, and the possibility that $(\rho \propto 1/e^N (N = 1,2,3,\ldots))$ at the resonance. Hitherto only the solution with $N = 1$ has been considered.

1. INTRODUCTION

The coupling of different MHD wave modes is of interest to many plasma physicists. The resonant coupling of fast and Alfvén waves is an important mechanism for heating laboratory plasmas [Hasegawa and Chen, 1976] and could play a significant role in heating the Sun's chromosphere. In a magnetospheric context it is thought that magnetic pulsations may be understood in terms of resonant wave coupling [Southwood, 1974; Chen and Hasegawa, 1974a].

Over recent years considerable efforts have been made to improve our understanding and modelling of coupling phenomena. Early studies concentrated upon steadily driven one-dimensional systems [Southwood, 1974; Chen and Hasegawa, 1974a] and required the treatment of resonant singularities in ordinary differential equations. These singularities occur at the location of the resonant field line. (The solution at the singularity can always be derived in the form of a series by employing the method of Frobenius [e.g., Bender and Orszag, 1978].)

More recent studies have tried to generalize the early results by considering a two-dimensional medium [Inhester, 1986; Southwood and Kivelson, 1986; Lee and Lysak, 1990; Mond et al., 1990; Wright, 1991; Wright, 1992a]. Some of the recent progress has been due to numerical solutions. In this paper we develop analytical methods more fully. Our aims are similar to those of Mond et al. [1990]. In their paper they adapted the series solution approach that had proved successful in early modelling. The more general medium considered by Mond et al., [1990] requires the treatment of resonant field lines in two spatial dimensions rather than one. The resulting equations are partial, rather than ordinary, differential equations (see also Pao [1978]).

In contrast to the ordinary differential equation (ODE) models, there is no standard method for deriving the solution of partial differential equations (PDEs) at a resonance (i.e., singularity). Mond et al., [1990] investigated a generalized series solution approach for the latter problem. Their calculation employed much physical insight, gained from earlier studies, to order the magnitudes of various perturbation quantities and enabled them to deduce the leading-order behavior of the singular solution - cf. Chen and Cowley [1989]. (Second-order resonance problems have two solutions, one regular and one singular.) General boundary conditions will produce a solution that can always be expressed as a sum of the regular and singular solutions. Thus the singular solution alone can not describe the details of resonance problems adequately.

The analysis presented in this paper provides a systematic procedure which will generate the singular and regular series solutions in the vicinity of the resonance. Moreover, we can achieve these results without recourse to invoking specific orderings of the magnitudes of perturbation quantities a priori. In addition, our approach does not require any restriction on the wave number [Wright, 1992a] nor do we require the density variation to be a separable function [e.g., Southwood and Kivelson, 1986].

Most recently, Schulze-Berge et al. [1992] have considered the existence of resonant surfaces within a uniform magnetic field permeating a three-dimensional density variation. Their calculation does, however, assume a given ordering of the perturbations.

The paper is structured as follows; In section 2 we derive the governing equations for a two-dimensional resonance problem in a uniform magnetic field (note that the density distribution varies in two directions). This system is identical to that considered in the "zero pressure" section of the paper by Mond et al. [1990], though we tend to follow the notation of, for example, Southwood [1974], Southwood and Kivelson [1986]. Section 3 derives a generalized Frobenius solution across the resonant field line. Section 4 demonstrates how the logarithmic terms in the Frobenius series may be continued across the resonant layer. Section 5 compares our results with those of previous studies, and section 6 concludes the paper.
2. Governing Equations

Throughout this paper we employ the linearized ideal MHD equations for a cold plasma in a uniform magnetic field $B = B_0$. The equilibrium plasma density $\rho_0(x, z)$ is invariant in the $y$ direction, but otherwise general. We assume that all perturbations have a dependence on $y$ and time $t$ of the form $\exp[i(\lambda y - \omega t)]$, the dependence upon $(x, z)$ is to be determined.

We shall introduce the Alfvén wave operator $\mathcal{L}$, which is defined as the differential operator

$$L \equiv B^2 \frac{\partial^2}{\partial z^2} + \mu_0 \rho_0(x, z) \omega^2.$$  

(1)

The momentum and induction equations may be combined with Ampère's and Ohm's laws to yield the following equations governing the transverse plasma displacement ($\xi_x, \xi_y$) and the compressional field component $b_z$:

$$\mathcal{L}_{\xi_x} = \frac{\partial b_z}{\partial x}$$  

(2a)

$$\mathcal{L}_{\xi_y} = i\lambda B b_z$$  

(2b)

$$\frac{b_z}{B} = -\frac{\partial \xi_x}{\partial x} - i\lambda \xi_y.$$  

(2c)

In earlier studies, where the density was independent of $z$, it was possible to derive a single ODE for $b_z$ and then use the conventional Frobenius treatment of the resonance. That is, the solution was developed as a power series in $x$, with the inclusion of a suitable logarithmic term to treat the singularity at the resonant field line [Bender and Orszag, 1978]. This is not possible in our more general model, however, we can simplify matters slightly by eliminating $b_z$ from the above equations

$$\mathcal{L}_{\xi_x} = \frac{1}{i\lambda} \frac{\partial \xi_y}{\partial x}$$  

(3a)

$$\mathcal{L}_{\xi_y} = -i\lambda B^2 \frac{\partial \xi_x}{\partial x} + \lambda^2 B^2 \xi_y.$$  

(3b)

We shall find it useful to expand the Alfvén wave operator $\mathcal{L}$ as a Taylor series in the $x$ coordinate [cf. Mond et al., 1990]. Without loss of generality we choose $x = 0$ to be the $x$-coordinate about which the expansion is performed. (The origin of $x$ can always be redefined to study the solution on different field lines.)

$$\mathcal{L} = \mathcal{L}_0 + x \mathcal{L}_1 + x^2 \mathcal{L}_2 + \ldots$$  

(4)

The operator $\mathcal{L}_0$ and the functions $\mathcal{L}_1, \mathcal{L}_2, \ldots$ are defined as

$$\mathcal{L}_0(\omega) = B^2 \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \rho_0(0, z)$$  

$$\mathcal{L}_1(\omega) = \mu_0 \omega^2 \left. \frac{\partial \rho_0}{\partial x} \right|_{x=0}$$  

$$\mathcal{L}_n(\omega) = \mu_0 \omega^2 \left. \frac{\partial^{n-1} \rho_0}{\partial x^n} \right|_{x=0}$$  

(5)

The operator $\mathcal{L}_0$ is the Alfvén wave operator for the field line located at $x = 0$. For the remainder of this paper we shall assume that the magnetic field lines are frozen in to dense perfectly conducting boundaries at $z = 0$ and $z = \ell$, so that $\xi_x, \xi_y$ and $b_z$ all vanish at $z = \ell$. This is the "box model" of Southwood [1974] but with density having a general dependence upon $x$ and $z$. Under these boundary conditions the wave operator is of the Sturm-Liouville form, and any solution to the operator may be written as a sum over the eigenmodes (or "harmonics") of the resonant field line. We shall denote the $n^\text{th}$ Alfvén wave eigenmode by the function $\phi_n(0, z)$, with $n = 1$ corresponding to the fundamental mode. The first argument of $\phi_n(0, z)$ denotes that this is an eigenmode of the field line at $x = 0$; we shall require eigenmodes of other field lines in section 4. (Unless stated explicitly, $\phi_n$ refers to the eigenmodes at $x = 0$, e.g., in sections 2.3 and the appendices.) The eigenmodes and their associated eigenfrequencies $\omega_n(x = 0)$ satisfy the relation

$$B^2 \frac{\partial^2 \phi_n}{\partial z^2} + \omega_n^2 \mu_0 \rho_0(0, z) \phi_n = 0;$$  

(6)

and it is simple to calculate the effect of operating upon any eigenfunction with $\mathcal{L}_0(\omega)$:

$$\mathcal{L}_0(\omega) \phi_n(0, z) = \mu_0 \rho_0(0, z) (\omega^2 - \omega_n^2) \phi_n(0, z).$$  

(7)

The condition for resonance may be regarded as being that a field line is driven at one of its natural eigenfrequencies, say the $r$th eigenfrequency $\omega_r$. Then the resonant eigenfunction $\phi_r$ will satisfy $\mathcal{L}_0(\omega_r) \phi_r(0, z) = 0$.

Because the eigenfunctions $\phi_n$ form a complete orthogonal set, any function $f(z)$ may be written as a sum over these functions weighted with appropriate coefficient $f_n$:

$$f(z) = \sum_{n=0}^{\infty} f_n \phi_n(0, z).$$  

(8)

The eigenmodes are orthogonal in the sense that when the product of two different modes is weighted by the density and integrated in $z$ along the resonant field line the result is zero,

$$\int_{(0,0)}^{(0,\ell)} \rho_0(\omega_n \phi_n)(\omega_n \phi_n) \rho_0 dz = 0, \quad n \neq n'.$$  

(9)

In the following sections we shall see that it is often necessary to invert the operator $\mathcal{L}_0$. For example, suppose we need to solve $\mathcal{L}_0 f(z) = g(z)$ for $f(z)$ when $g(z)$ is a known function. Writing $f$ as the sum in (8), the inversion problem amounts to determining the set of coefficients $\{f_n\}$. Employing (7) and (8), $\mathcal{L}_0 f(z) = g(z)$ may be written

$$\mu_0 \rho_0(0, z) \sum_{n=0}^{\infty} (\omega^2 - \omega_n^2) \phi_n f_n = g(z).$$  

(10)

The coefficient of any mode may be isolated by multiplying (10) by the relevant mode, and integrating along the $x = 0$ field line. Recalling the orthogonal property (9) we find

$$\mu_0 (\omega^2 - \omega_n^2) \rho_0 (0, z) \phi_n f_n = (g(z) \phi_n)$$  

(11)

and hence, provided $\omega_n \neq \omega$, \[ f_n = \frac{(g(z) \phi_n)}{\mu_0 (\omega^2 - \omega_n^2) \rho_0 (0, z) \phi_n \phi_n}. \]

(12)

This defines the solution $f = g$, and hence the inverse operator $\mathcal{L}_0^{-1}$, in the case when $\omega$ is not equal to any of the eigenfrequencies of the $x = 0$ field line. In this paper, however, we are interested principally in the resonant case, when the driving frequency $\omega$ is equal to $\omega_r$ for some $r$. In this case, it is evident from (11) that $f_r$ cannot be determined, although all of the other coefficients are still defined by (12). Indeed, when considering the resonant coefficient ($n \equiv r$) we see that the function $g$ cannot be completely
arbitrary if a solution \( f \) is to exist (see equation (11)): a solution \( f \) of \( \mathcal{L}_0 f(z) = g(z) \) only exists if \( g(z) \) satisfies the solvability condition \( [SC] \)
\[
\mathcal{L}_0 f(z) = 0 .
\] (13)
The solution \( f(z) \) obtained by inverting \( \mathcal{L}_0 \) has an unspecified amount of the resonant eigenfunction present. Thus the inversion of \( \mathcal{L}_0 f(z) = g(z) \), where \( g \) satisfies the SC (13) and \( \omega \) is equal to the natural frequency \( \omega_r(0) \), yields
\[
f(z) = \sum_{n \neq r} \frac{\langle g(z) \phi_n \rangle}{(\omega^2 - \omega_n^2)(\rho(0,z)\phi_n\phi_n)} \phi_n(0,z) + \gamma \phi_r ,
\] (14)
where \( \gamma \) is an arbitrary constant. We shall write the right-hand side of equation (14) as being identically equal to
\[
\mathcal{L}_0^{-1} g + \gamma \phi_r ,
\] (15)
thus defining the operator \( \mathcal{L}_0^{-1} \) in the resonant case.

3. Generalized Frobenius Solution

In this section we outline the generalized Frobenius procedure for solving equations (3). To illustrate the method we develop the nonresonant solution in some detail and then indicate the main differences in the method when the \( x = 0 \) field line is driven at a resonant frequency. Full details are deferred to Appendices A and B.

Having expanded the operator \( \mathcal{L} \) (equation [5]) as a series in \( x \) we seek solutions \( c_0 \) and \( c_1 \) to equations (3) by similarly writing these functions as series expansions in \( x \). Guided by the Frobenius solution to ordinary differential equations, we allow for the possibility of a logarithmic term and therefore posit the following ansatz
\[
x = x^\sigma \sum_{n=0}^\infty a_n(z)x^n + x^\sigma \ln x \sum_{n=0}^\infty c_n(z)x^n \quad (16a)
\]
\[
x = x^\sigma \sum_{n=0}^\infty b_n(z)x^n + x^\sigma \ln x \sum_{n=0}^\infty d_n(z)x^n . \quad (16b)
\]
Here the \( a_n, b_n, c_n \) and \( d_n \), which are to be determined, are functions of \( z \), in contrast with the ODE case where they would be constants. In order to make \( \sigma \) (which thus far is also unknown) definite, we may assume that at least one of \( a_0, b_0, c_0 \) and \( d_0 \) is not identically zero. Moreover, the functions \( a_n, b_n, c_n \) and \( d_n \) must satisfy the boundary conditions at \( z = 0 \) and \( z = 1 \). Upon substituting the expansions (16) and (4), into the equations (3), we obtain two equations in both of which the terms in \( x^{n+\sigma} \) and \( x^{n+\sigma}\ln x \) must separately balance. From the lowest-order terms \( \sigma \) can be determined (cf. the indicial equation in the application of the Frobenius method to ODEs): the algebraic details may be found in Appendix A. We distinguish two cases. In the nonresonant case, by which we mean that the externally imposed driving frequency \( \omega \) matches none of the eigenfrequencies \( \omega_n \) of the \( x = 0 \) field line, only \( \sigma = 0 \) is permissible. In the case when \( \omega = \omega_r \) for some \( r \), which we shall call the resonant case, both \( \sigma = 0 \) and \( \sigma = -1 \) are possible. In both resonant and nonresonant cases, all the coefficients \( c_n \) and \( d_n \) are identically zero if \( \sigma = 0 \) and so there is no logarithmic term: hence this is the regular solution. The \( \sigma = -1 \) solution contains a logarithmic term and hence we call it the singular solution.

Nonresonant Case

To illustrate the way in which the Frobenius solution is developed, let us consider the nonresonant case. Then, as has already been stated and as is proved in Appendix A, \( \sigma = 0 \) and the \( c_n \) and \( d_n \) are identically zero. Thus equations (16) become
\[
\xi_x = \sum_{n=0}^\infty a_n x^n , \quad \xi_y = \sum_{n=0}^\infty b_n x^n .
\]
Substituting these expansions and the expansion (4) into equations (3) gives
\[
\sum_{n=0}^\infty \left( \mathcal{L}_0 a_n + x\mathcal{L}_1 a_n + x^2\mathcal{L}_2 a_n + \ldots \right) x^n = \frac{1}{i\lambda} \sum_{n=0}^\infty \left[ n\mathcal{L}_0 b_n + (n+1)x\mathcal{L}_1 b_n \right] + \frac{1}{i\lambda} \sum_{n=0}^\infty \left[ n\mathcal{L}_0 b_n + (n+2)x^2\mathcal{L}_2 b_n + \ldots \right] x^{n-1} \quad (17a)
\]
\[
\sum_{n=0}^\infty \left( (\mathcal{L}_0 - \lambda^2 B^2)b_n + x\mathcal{L}_1 b_n + x^2\mathcal{L}_2 b_n + \ldots \right) x^n = -i\lambda B^2 \sum_{n=0}^\infty n a_n x^{n-1} . \quad (17b)
\]
For the lowest-order terms \( (x^0) \) to balance in each equation requires that
\[
\mathcal{L}_0 a_0 = \frac{1}{i\lambda} \left( \mathcal{L}_0 b_1 + \mathcal{L}_1 b_0 \right) , \quad \left( \mathcal{L}_0 - \lambda^2 B^2 \right) b_0 = -i\lambda B^2 a_1 ;
\]
\[
\mathcal{L}_0 a_1 + \mathcal{L}_1 a_0 = \frac{2}{i\lambda} \left( \mathcal{L}_0 b_2 + \mathcal{L}_1 b_1 + \mathcal{L}_2 b_0 \right) , \quad \left( \mathcal{L}_0 - \lambda^2 B^2 \right) b_1 + \mathcal{L}_1 b_0 = -2i\lambda B^2 a_2 ; \quad (19)
\]
and so on. In this solution \( a_0 \) and \( b_0 \) are arbitrary functions (except that they must satisfy the boundary conditions). Since the governing equations are linear, we may consider the two solutions \( a_0 \neq 0, b_0 = 0 \) and \( b_0 \neq 0, a_0 = 0 \) separately. Recalling that in the nonresonant case, the inverse of \( \mathcal{L}_0 \) is well-defined (equation (12), it is readily apparent that the pairs of equations (18), (19) (and so on) determine the functions \( a_n \) and \( b_n \) for \( n = 1, 2, \ldots \). Thus one deduces (see Appendix A for details) that the solution in the nonresonant case is a linear combination of the solution generated by \( a_0 \) and the solution generated by \( b_0 \):
\[
\xi_x = a_0 - \frac{1}{i\lambda B^2} \left( \mathcal{L}_0 - \lambda^2 B^2 \right) b_0 x
\]
\[
+ \left[ -\frac{1}{2B^2} \left( \mathcal{L}_0 - \lambda^2 B^2 \right) a_0 + \frac{1}{2} i\lambda \mathcal{L}_0^{-1} \mathcal{L}_1 b_0 \right] x^2
\]
\[
+ O(x^3) \quad (20a)
\]
\[
\xi_y = b_0 + \left[ i\lambda a_0 - \mathcal{L}_0^{-1} \mathcal{L}_1 b_0 \right] x
\]
\[
- \left[ \frac{1}{i\lambda} \mathcal{L}_0^{-1} \mathcal{L}_1 a_0 + \left\{ \mathcal{L}_0^{-1} \mathcal{L}_2 - \left( \mathcal{L}_0^{-1} \mathcal{L}_1 \right)^2 \right\} b_0
\]
\[
+ \frac{1}{2B^2} \left( \mathcal{L}_0 - \lambda^2 B^2 \right) b_0 \right] x^2
\]
\[
+ O(x^3) , \quad (20b)
\]
where \( a_0 \) and \( b_0 \) are arbitrary functions of \( x \). Furthermore, from equation (2b),
Resonant Case

Regular Solution. Henceforth we restrict attention to the resonant case ($\omega^2 = \omega_r^2$ for some $r$). Let us consider first the regular solution ($\sigma = 0$). Equations (19) and (20) still hold: the difference is that now $L_0$ is invertible only if the solvability condition is satisfied. Once again $a_0$ is arbitrary; and $b_0$ is arbitrary except for a single restriction. The first of equations (18) can be rewritten as

$$L_0(\lambda \lambda a_0 - b_1) = L_1 b_0 ;$$

and it follows from the solvability condition (13) that this is only solvable for $b_1$ if $(\phi_r L_1 b_0) = 0$. Once again the solutions for $a_0 \neq 0$, $b_0 = 0$ and $b_0 \neq 0$, $a_0 = 0$ can be considered separately and then combined in any linear combination. For example, in the case $b_0 = 0$ it follows from equation (21) that

$$b_1 = i \lambda a_0 + \gamma \phi_r ,$$

where the constant $\gamma$ is (as yet) undetermined. The constant is determined by going to the next order: the solvability condition applied to the first of equations (19) gives

$$(\phi_r L_1 b_1) = \frac{1}{2} i \lambda (\phi_r L_1 a_0)$$

and hence, using equation (22),

$$\gamma = \frac{-i \lambda (\phi_r L_1 a_0)}{2(\phi_r L_1 \phi_r)} .$$

Continuing this procedure (see Appendix A for details), the regular solution in the resonant case is

$$\xi_r = a_0 - \frac{1}{\chi^2 B_0^2} (L_0 - \lambda^2 B^2) b_0 x + O(x^2)$$

and

$$\xi_r = b_0 + \frac{1}{\lambda^2 B_0^2} \left[ i a_0 (\lambda \phi_r - L_0 b_0) + \gamma_1 \phi_r \right] x + O(x^2) ,$$

with

$$\beta_1 = \frac{-1}{2} \lambda (\phi_r L_1 a_0)/(\phi_r L_1 \phi_r) ,$$

$$\gamma_1 = (\phi_r L_1 \gamma_0 - b_0)/(\phi_r L_1 \phi_r) .$$

Also, from equation (2b),

$$b_2 = \frac{1}{\lambda^2 B_0^2} L_0 b_0 + \frac{1}{\lambda^2 B_0^2} a_0 x + O(x^2) .$$

The functions $a_0(x)$ and $b_0(x)$ are arbitrary except that $(\phi_r L_1 b_0)$ must be zero.

Singular Solution. The singular solution ($\sigma = -1$) is a little more complicated algebraically because the coefficients $c_n$ and $d_n$ of the logarithmic terms are no longer identically zero. The details are given in Appendix A. Substituting the expansions (16) in equations (3) (cf. equations (A3)), and balancing terms of order $x^{\sigma-1}$, one obtains

$$\sigma a_0 + c_0 = 0 , \quad \sigma L_0 b_0 + L_0 d_0 = 0 ;$$

and balancing the $x^{\sigma-1} \ln x$ terms yields

$$\sigma a_0 = 0 , \quad \sigma L_0 d_0 = 0 .$$

Since $\sigma \neq 0$ for the singular solution, these equations imply that $a_0 = c_0 = 0$ and $L_0 b_0 = L_0 d_0 = 0$, i.e., $b_0$ and $d_0$ are multiples of the resonant eigenfunction $\phi_r$. But at the next order one finds that

$$L_0 d_1 + L_1 d_0 = 0 ;$$

from which (using the solvability condition) one finds that $d_0 = 0$. Hence the only nonzero coefficient at lowest order is $b_0 = \beta_0 \phi_r$ (where $\beta_0$ is an arbitrary constant). From the remaining equations at this order, and proceeding to higher orders, one finds that

$$\xi_r = -i \lambda \beta_0 \phi_r \ln x + O(x, x^2 \ln x)$$

and

$$\xi_r = \beta_0 \phi_r \frac{1}{x} - \beta_0 (\phi_r L_2 \phi_r)/(\phi_r L_1 \phi_r) \phi_r$$

+ $\frac{1}{2} \chi^2 \beta_0 \phi_r x \ln x + O(x, x^2 \ln x)$

Thus the logarithmic nature of the singularity is in accord with the results of previous studies. It is possible to derive recursion formulae for the coefficients in the generalized Frobenius solutions. The details are given in Appendix B.

4. CONTINUING THE SOLUTION ACROSS THE SINGULARITY

To complete the singular solution it is necessary to determine how to continue the solution across the singularity. One way to do this is to allow the boundaries (at $z = 0, \ell$) to be weakly absorptive. In our approach, which is based upon that of Mond et al. [1990], we consider that the boundaries are not absorptive but rather that the system is weakly driven, so that the time dependence is $\exp(-i \omega t)$, where $\omega = \omega_r + i \omega_l$ ($\omega_r$ real, $\omega_l > 0$). The growth time, which is of order $2 \pi/\omega_l$, is assumed to be much greater than any Alfvén or fast mode transit time of the system. Here $\omega_r$ is the real driving frequency that was written in previous sections without a subscript. Thus $\omega_R$ is equal to the square of the $r$th eigenfrequency of the $\sigma = 0$ field line, viz, $\omega_R^2 = \omega_r^2(0)$.

When $\omega$ is complex, there is no longer an exact resonance at $x = 0$ (or any other real $x$). In a formal sense, the resonance now lies instead at a complex position $x_{res} = 0 + i x_l$. Of course, $x_l$ is a function of $\omega_l$. (Note that at this stage $x_l$ may be complex.) We seek to determine the dependence of $x_l$ upon $\omega_l$, most importantly its sign.

In this section we shall write

$$B^2 \frac{\partial^2}{\partial x^2} + \mu_0 \rho_0 (x, z) \omega^2 \equiv L(\omega, x)$$

(cf. equation (1)); $L$ has an implicit $z$ dependence, which should be understood. As before, the operator $L$ can be expanded in a Taylor expansion about the resonant position:

$$L(\omega, x) = L(\omega, x_{res}) + (x - x_{res}) L_1(\omega, x_{res}) + \ldots$$

Now $L(\omega, x_{res})$ can in turn be expressed as a Taylor expansion about the original resonance.
\[ \mathcal{L}(\omega, \xi_{\text{res}}) = \mathcal{L}(\omega, 0) + i\xi_1\mathcal{L}_1(\omega, 0) + i\omega I \frac{\partial \mathcal{L}}{\partial \omega}(\omega, 0) + \ldots . \]  
(29)

The choice of driving frequency determines both the position \( x_{\text{res}} \) of the resonance and the form of the resonant eigenfunction \( \phi_{\text{res}} \), where

\[ \mathcal{L}(\omega, x_{\text{res}})\phi_{\text{res}} = 0. \]  
(30)

We can write \( \phi_{\text{res}} \) in terms of the eigenfunction \( \phi_r \) at \( x = 0 \) with eigenfrequency \( \omega_r \) as

\[ \phi_{\text{res}} = \phi_r + \Delta \phi_{\text{res}}. \]  
(31)

Using expression (29) up to first order in small quantities for \( \mathcal{L}(\omega, x_{\text{res}}) \) and expression (31) for \( \phi_{\text{res}} \), equation (30) yields

\[ ix_1\mathcal{L}_1(\omega, 0)\phi_r + i\omega I \frac{\partial \mathcal{L}}{\partial \omega}(\omega, 0) + \mathcal{L}(\omega, 0)\Delta \phi_{\text{res}} = 0, \]  
(32)

using \( \mathcal{L}(\omega, 0)\phi_r = 0 \). Multiplying by \( \phi_r \) and integrating with respect to \( x \) at \( x = 0 \), we find

\[ x_1 = -\omega I \frac{\phi_r}{\omega} \frac{\partial \mathcal{L}}{\partial \omega}(\phi_r \mathcal{L}_1(\omega, 0)\phi_r), \]  
(33)

where the operators \( \partial \mathcal{L}/\partial \omega \) and \( \mathcal{L}_1 \) are evaluated at \( \omega = \omega_r \) and \( x = 0 \). Note that the final step is possible because \( \Delta \phi_{\text{res}} \) satisfies the same perfectly reflecting boundary conditions as \( \phi_r \) at \( z = 0, l \). If the boundaries were weakly absorptive, there would be an additional term in equation (33). For our differential operator, equation (33) may be rewritten as

\[ x_1 = -\frac{2\omega I}{\omega} \frac{\phi_r}{\omega} \frac{\partial \mathcal{L}}{\partial \omega}(\phi_r \mathcal{L}_1(\omega, 0)\phi_r); \]  
(34)

where \( \rho \) and \( \partial \rho/\partial \omega \) are evaluated at \( x = 0 \). Note that equation (34) implies that \( x_1 \) is real to this order.

The Frobenius series solutions in the weakly driven case are the same as before, except that instead of being expansions of powers of \( x - 0 \) and its logarithm, they are expansions in powers of \( (x - \xi_{\text{res}}) \) and its logarithm. Consider then the argument of \( \ln(z - izi) \) as one goes through the solution across the resonance in the case of a real driving frequency. The above energy flux is positive definite: thus the resonance absorbs energy, which is in accord with the original interpretation (2b). Multiplying this relation (35) and (37) may be combined to yield an expression for the power absorbed over the length of the resonant field line, per unit length in the \( y \) direction

\[ \langle S_x(0^-) \rangle - \langle S_x(0^+) \rangle = \pi \omega^2 |\beta_0|^2 (\phi_r \mathcal{L}_1 \phi_r) \text{sign}[\xi_1], \]  
(35)

by equations (26). At the end of this section we show that the difference in the above energy fluxes is positive, so the resonance absorbs energy. (The regular solution gives a continuous contribution to the Poynting flux and so does not affect this calculation.)

In the previous studies of ODE resonance problems [Southwood, 1974], it is clear that the change in phase of the argument of the logarithm depends solely upon the gradient of the rth Alfvén eigenfrequency at the resonance. The choice of sign of the phase change means that the resonance absorbs energy, rather than radiates energy. It is not evident from our more general analysis (see (33) and (34)) how \( z_1 \) depends on \( dw^2(x)/dx \). For example, is it possible to specify an \( \mathcal{L} \) which will yield an arbitrary eigenfrequency gradient across the resonance? The factor \( \langle \phi_r \mathcal{L}_1 \phi_r \rangle \) enters many of our expressions (e.g., equation (33)), and it is worth developing an interpretation for it. Let the \( r \)th eigenmode and eigenfrequency satisfy equation (6) at \( x = 0 \). Now consider the change in eigenmode and eigenfrequency on an adjacent field line at \( x = \delta x \) \((\delta x \text{ real}) \). The change to the density, eigenfrequency and eigenmode relative to those on \( x = 0 \) are \( \delta \rho_0 = \delta x (\partial \rho_0(\omega, x)/\partial \omega), \delta \omega_0^2 = \delta x (\partial \omega_0^2/dx) \) and \( \delta \phi_r = \delta x (\partial \phi_r(x, z)/\partial \omega) \) respectively, all the derivatives being evaluated at \( x = 0 \). Substituting these changes in to a Taylor series expansion of \( \mathcal{L} \phi_r = 0 \) about \( x = 0 \) we find the terms linear in \( \delta x \) satisfy

\[ \mathcal{L}_0 \frac{\partial \phi_r}{\partial x} \bigg|_{x=0} + \left[ \mathcal{L}_1 + \mu_0 \rho_0(0, z) \frac{\partial \omega^2}{\partial z} \bigg|_{x=0} \right] \phi_r(0, z) = 0. \]  
(36)

Multiplying equation (36) by \( \phi_r \) and integrating along the background field line \( x = 0 \) we find the simple relation

\[ \frac{\partial \omega^2}{\partial x} \bigg|_{x=0} = \frac{1}{\mu_0} \frac{\partial}{\partial x} \left[ \phi_r \mathcal{L}_1 \phi_r \right]. \]  
(37)

Recalling the definition of \( \mathcal{L}_1 \) it is clear from (34) and (36) that \( z_1 = \omega_1/(\partial \omega_0/dx) \). Thus, as one would anticipate, the sense of the phase change of the log terms across the resonance is determined by the gradient of the resonant eigenfrequency.

Note that the signs of \( z_1, \omega_1 \) and \( dw_0/\partial x \) are related by \( \text{sign}(dw_0/\partial x) = \text{sign}[\xi_1] \), since \( \omega_1 \) is by assumption positive. Employing this relation (35) and (37) may be combined to yield an expression for the power absorbed over the length of the resonant field line, per unit length in the \( y \) direction

\[ \langle S_x(0^-) \rangle - \langle S_x(0^+) \rangle = \pi \omega^2 |\beta_0|^2 (\phi_r \rho_0 \phi_r) \left( \frac{\partial \omega_0}{\partial x} \right) \]  
(38)

The above energy flux is positive definite: thus the resonance absorbs energy, which is in accord with the original one-dimensional calculations [Chen and Hasegawa, 1974b; Southwood, 1974].

5. DISCUSSION

It is instructive to compare our results with those of previous workers, who have in the main concentrated on the singular solution. Southwood and Kivelson [1986] and Wright [1992a,b] have viewed the magnetic field perturbation \( b_0 \) as the driving system given by equations (2a, b). From their solutions in (25c) and (26c) it is evident that the leading-order of \( b_0 \) is \( O(\xi^5) \), which we shall denote as \( b_0(z) \). We can relate the leading-order singular solution for \( \xi_0 \), namely \( \beta_0 \phi_r(z)/\omega_1 \phi_r(z)/\omega_1 \), by considering the expanded form of (2b). Multiplying by \( \phi_r \) and integrating in \( z \) we find

\[ \langle S_x(0^-) \rangle - \langle S_x(0^+) \rangle = \pi \omega^2 |\beta_0|^2 (\phi_r \mathcal{L}_1 \phi_r) \left( \frac{\partial \omega_0}{\partial x} \right) \]  
(38)
property affirms previous results [Southwood and Kivelson, 1986; Chen and Couley, 1989; Wright, 1991]. It is instructive to note here that while the singular solution dominates the displacement $\xi$ near the resonance and has the $z$ dependence of the resonant eigenfunction, there is no such restriction on the $z$ dependence of the singular solution, and is evidently sensitive to the variation of $b_0$ along the resonant field line. This property affirms previous results [Southwood and Kivelson, 1986; Chen and Couley, 1989; Wright, 1991].

The second equality uses equation (37). Note how the parameter $\beta_0$ depends upon the overlap integral of $b_0$ and the resonant eigenfunction; $\beta_0$ represents the amplitude of the singular (or resonant) response, and is evidently sensitive to the variation of $b_0$ along the resonant field line. This property affirms previous results [Southwood and Kivelson, 1986; Chen and Couley, 1989; Wright, 1991]. It is instructive to note here that while the singular solution dominates the displacement $\xi$ near the resonance and has the $z$ dependence of the resonant eigenfunction, there is no such restriction on the $z$ dependence of the singular solution, and is evidently sensitive to the variation of $b_0$ along the resonant field line. This property affirms previous results [Southwood and Kivelson, 1986; Chen and Couley, 1989; Wright, 1991].

The expresion above demonstrates many features found in previous studies, such as the dependence on $\lambda^2$, the square of the overlap integral representing the efficiency of coupling, and the inverse dependence upon $\partial \sigma/\partial z$ (see, for example, equation (18) of Wright [1992a]).

Some previous studies have been able to deduce the leading behavior of $\xi_0$ and $\xi_0$ for the singular solution. Although our method enables us to determine the behavior at all orders for these displacements, we list the leading order terms here to facilitate comparison with existing results. From equations (26) and (39) we see

\[
\frac{\partial}{\partial x} (\phi_0(x) \phi_0(x)) = \frac{-i \lambda B}{2 \mu_0 \omega_r} \left( \frac{d \omega_r}{dx} \right)_{x=0} \phi_0(0, x) \ln(x) \quad (41a)
\]

\[
\xi_0(x) \approx \frac{-i \lambda B}{2 \mu_0 \omega_r} \left( \frac{d \omega_r}{dx} \right)_{x=0} \phi_0(0, z) \ln(x) \quad (41b)
\]

\[
\phi_{0,0} \approx \frac{1}{b_0} \quad (41c)
\]

Note that these expressions are for quite arbitrary density, $\rho_0(x, z)$. Southwood and Kivelson [1986] produced a similar expression to our (41b) when they imposed the simplification of having a density distribution that was a separable function of $x$ and $z$. To compare with their results we shall write $\rho_0(x, z) = \rho_2(x) \rho_z(z)$. From the structure of the eigenvalue equation it follows that the $n$th eigenfrequency (which will be a function of $x$) will be determined by the variation of $\rho_2(x)$; The quantity $\omega_n^2 \rho_2$ is the pseudo-eigenvalue for the operator $B^2 \partial^2/\partial z^2 + [\omega_n^2 \rho_2] \mu_0 \rho_2(z)$ and is independent of $x$. Thus, from

\[
\omega_n^2(x) \rho_2(x) = \text{const} \quad (42)
\]

we deduce that

\[
\frac{d \omega_n^2}{dx} = -\frac{\omega_n^2}{\rho_z} \frac{d \rho_z}{dx} \quad (43)
\]

For the resonant eigenmode we shall require $\omega^2 = \omega_n^2(0)$. Recalling the definition of the operator $L_1$ (equation[5]) and employing the relation (43), $L_1$ may be expressed in the form

\[
L_1 = -\mu_0 \rho_z(x) \rho_2(x) \left[ \frac{d \omega_n^2}{dx} \right]_{x=0} = 0 \quad (44)
\]

By defining $[d \omega_n^2/dx]_0 = \omega_n^2(x) - \omega_n^2(0) / x$ in the limit $x \to 0$, we find

\[
L_1 = -\mu_0 \rho_z(x) \rho_2(x) \frac{\omega_n^2(x) - \omega_n^2(0)}{x} \quad (45)
\]

and equation (41b) gives the leading behavior of $\xi_0$ as

\[
\xi_0 \approx \frac{i \lambda B}{\mu_0 \rho_z(0)} \frac{\phi_0(0, x)}{\phi_0(0, z) \phi_0} \left[ \omega_n^2 - \omega_n^2(x) \right] \quad (46)
\]

The above expression is identical to equation (34) of Southwood and Kivelson [1986], if we assume that their normalised eigenfunctions satisfy $\phi_0(x, z) = 1$. Thus we are able to recover the existing results of Southwood and Kivelson [1986] by taking the appropriate limit of our more general analysis.

Note that the amplitudes of the singular solution fields in equations (41a) and (41b) are inversely proportional to $[d \omega_n^2/dx]_0$, suggesting that our singular solution analysis breaks down when

\[
\frac{d \omega_n^2}{dx} \biggr|_{x=0} = 0 \quad (47)
\]

This is indeed the case. When the above relation is satisfied we conclude that not only is the field line at $x = 0$ resonant, but field lines close to $x = 0$ appear, at lowest order, to be resonant too. It should be noted that the proof of $\sigma = -1$ for the singular solution (Appendix A) relied upon the property $\phi_0 L_1 \phi_0 = 0$ when we invoked the solvability condition. When considering the situation in which (47) is valid, however, equation (37) implies that $\langle \phi_0 L_1 \phi_0 = 0 \rangle$ and hence we cannot assume that $\sigma = -1$ for the singular solution. The method set down in Appendix A might be extended to this case. However, it is instructive instead to deduce the new singular index value in the following manner, provided we assume that the lowest-order behavior of $b_0$ is still of order $1$ (say, $b_0(x)$, again). Expanding $L$ according to (5), and integrating the product of $\phi_0$ with equation (26) along the field line at $x = 0$ we find balancing powers of $x$ will arise from the following terms

\[
\phi_0(L_0 + xL_1 + x^2L_2 + \ldots)(\beta_0 \phi_0 x^\sigma + b_1 x^{\sigma+1} + b_2 x^{\sigma+2} + \ldots) = i \lambda B \left[ b_0(x) \phi_0 \right] \quad (48)
\]

where we have used the property that $b_0 = \beta_0 \phi_0(0, x)$ if $\sigma = 0$ (see (A5)). Evidently, when $\phi_0 L_1 \phi_0 \neq 0$, the term that is $O(1)$ on the right-hand side of equation (48) is balanced by a term on the left-hand side which requires $\sigma = -1$. However, when $\phi_0 L_1 \phi_0 = 0$, balancing terms on
both sides requires that \( \sigma = -2 \); but then only provided that \( (\phi_{\nu}[L_{1}b_{1} + L_{2}b_{0}]) \) is nonzero. If this factor were zero, we would have to try and balance terms with \( \sigma = -3 \).

In the case where equation (47) holds and the singular solution has an index of \(-2\), we find the leading order behavior of \( \xi_{s} \) to be

\[
\xi_{s} \approx \beta_{0} \frac{\phi_{\nu}}{x}. \tag{49}
\]

while from equation (2c) the behavior of \( \xi_{s} \) is

\[
\xi_{s} \approx i\lambda \beta_{0} \frac{\phi_{\nu}}{x}. \tag{50}
\]

For example, such a circumstance would arise in the box model [Southwood, 1974] if \( \partial \phi_{\nu} / \partial z \) vanished at the resonant field line (provided \( \partial \phi_{\nu} / \partial z \) also did not vanish). In that case, the solutions for \( \xi_{s} \), \( \xi_{u} \) and \( b_{s} \) contain no logarithmic terms and hence no Poynting flux is absorbed by the layer. It is easy to investigate the character of the singular solution of the one-dimensional box model (governed by an ODE).

More generally, we find that if the first nonzero derivative of the density is the \( N \)th, then there are no logarithmic terms in the singular solution when \( N \) is even, and no energy absorption at the resonance (\( \xi_{s} \sim 1/2^{N_{1}} \); \( \xi_{u} \sim 1/2^{N_{2}} \)). When \( N \) is odd there is a logarithmic term in the singular solution and a net absorption of energy by the resonant field lines. The singular solution has an index of \(-2\), \( \xi_{u} \sim 1/2^{N_{1}} \ldots + \ln x \); \( \xi_{s} \sim 1/2^{N_{2}} \). It would appear that energy absorption at a resonance is dependent upon the presence of a monotonic change in natural Alfvén frequency across the resonance.

6. CONCLUSIONS

We have presented a systematic method for determining the regular and singular solutions at the singularity in a partial differential equation describing the resonant excitation of Alfvén waves. Future work will generalize this analysis and enable us to consider more general magnetoplasmas (e.g., curl-free magnetic fields, force-free magnetic fields, or warm plasmas).

Previous authors [Southwood and Kivelson, 1986; Chen and Cowley, 1989; Wright, 1991; Wright, 1992a] have demonstrated the importance of the “overlap integral” \( b_{s} \phi_{s} \) which describes how efficiently the fast mode may drive a resonance. This quantity appears quite naturally in our series solution. It may be noted that our derivation does not impose any of the simplifications of earlier studies: namely that \( p_{0} \) is a separable function of \( x \) and \( z \), or that the wave number \( \lambda \) is small.

Note added in proof. During publication of this paper an article by Hansen and Goertz (Phys. Fluids 4, 2713, 1992) appeared which called into question the validity of the “field line resonance” (FLR) expansion. This refers to the method of solution employed by, for example, Chen and Cowley [1989] and Mond et al. [1990]. Hansen and Goertz argue that the FLR expansion leads to an inconsistent solution. Hansen and Goertz substantiate their claim by using a similar generalised Frobenius series to ours (their equation (31)); we note, however, that in that equation they omit terms – presumably because they are regular – which nonetheless should be included because they would contribute in later equations at the same order as terms which they have considered.

Our present paper solves explicitly for the coefficients in this series, which Hansen and Goertz did not do. We demonstrate the existence of a singular solution with leading order variation along the resonant field line of the resonant eigenfunction, in contradiction to the conclusions of Hansen and Goertz.

APPENDIX A: GENERALIZED FROBENIUS SOLUTIONS FOR THE REGULAR AND SINGULAR SOLUTIONS

For completeness we restate the governing coupled equations that we wish to solve:

\[
\frac{\partial \xi_{x}}{\partial x} = -\frac{1}{i\lambda B^{2}} \left( L \xi_{y} - \lambda^{2} B^{2} \xi_{s} \right) \tag{A1a}
\]

\[
\frac{\partial \xi_{y}}{\partial x} = i\lambda L \xi_{x} \tag{A1b}
\]

(cf. equations (3)). We assume that the solution can be written as an expansion of the form

\[
\begin{aligned}
\xi_{x} &= x^{\sigma} \sum_{n=0}^{\infty} \left( \frac{a_{n}(x)}{b_{n}(x)} \right) x^{n} + x^{\sigma} \ln x \sum_{n=0}^{\infty} \left( \frac{c_{n}(x)}{d_{n}(x)} \right) x^{n} \\
\xi_{y} &= \frac{\ln \left( \frac{\sigma_{0} a_{0} x^{\sigma-1} + (\sigma + 1) a_{1} x^{\sigma} + \ldots}{(\sigma + 1) a_{1} x^{\sigma} + (\sigma + 2) a_{2} x^{\sigma+1} + \ldots} \right)}{(\sigma a_{0} x^{\sigma} + (\sigma + 1) a_{1} x^{\sigma+1} + \ldots)}
\end{aligned} \tag{A2}
\]

(cf. equations (16)).

For definiteness in determining \( \sigma \), at least one of \( a_{0}, b_{0} \) and \( d_{0} \) must be nonzero. Substituting the expansion (A2) into equations (A1) yields

\[
\begin{aligned}
-x \lambda B^{2} &\left[ \sigma a_{0} x^{\sigma-1} + (\sigma + 1) a_{1} x^{\sigma} + \ldots \right] \\
&+ (\sigma + 2) a_{2} x^{\sigma+1} + \ldots
\end{aligned} \tag{A3a}
\]

\[
\begin{aligned}
&+ \ln x \left[ (\sigma a_{0} x^{\sigma-1} + (\sigma + 1) a_{1} x^{\sigma} + \ldots) \\
&+ (\sigma + 2) (a_{2} x^{\sigma+1} + \ldots) \right]
\end{aligned} \tag{A3b}
\]

In the nonresonant case, \( L_{0} f = 0 \) implies that \( f = 0 \), and \( L_{0} f = g \) can always be inverted to infer \( f \) in terms of \( g \). In the resonant case, the other hand, there exists a resonant eigenfunction \( \phi_{s} \) such that \( L_{0} \phi_{s} = 0 \). In this case, the equation \( L_{0} f = g \) is not always solvable for \( f \), as has already been noted in section 2: in fact it is only solvable if \( \phi_{s} = 0 \), in which circumstance we adopt the definitions (14) and (15) for the inverse operator.

Throughout this paper we have to invert equations of the form \( L_{0} f = L_{1} \gamma \phi_{s} = 0 \). The solvability condition, equation (13), requires that

\[
\gamma(\phi_{s}, L_{1} \phi_{s}) = 0. \tag{A4}
\]

At the end of section 4 it is shown that \( \langle \phi_{s}, L_{1} \phi_{s} \rangle \neq 0 \) is equivalent to stating \( [d\omega_{2}(x)/dx]_{0} \neq 0 \), which we would expect to be true for most media. The consequences of \( [d\omega_{2}(x)/dx]_{0} = \langle \phi_{s}, L_{1} \phi_{s} \rangle = 0 \) are discussed further in sec-
tion 5. For the principal calculation in this paper we shall assume that \( \langle \phi, L_1 \phi \rangle \neq 0 \); hence equation (A4) implies that \( \gamma = 0 \).

**Allowable values of the index \( \sigma \)**

Balancing \( x^{\sigma-1} \) terms and \( x^{\sigma-1} \ln x \) terms in equations (A3) requires that

\[
\sigma a_0 = 0, \quad \sigma L_0 b_0 = 0, \quad c_0 = 0 \quad \text{and} \quad L_0 d_0 = 0.
\]

**Nonresonant case.** In this case, \( L_0 \) is invertible. Thus, if \( \sigma \neq 0 \), equations (A5) imply that \( a_0 = c_0 = d_0 = 0 \), which contradicts the assumption that these functions are not all identically zero. Hence in the nonresonant case \( \sigma \) must be zero.

**Resonant case.** Suppose that \( \sigma \neq 0 \) and \( \sigma 
eq -1 \). Equations (A5) then imply that \( a_0 = c_0 = 0 \) and \( L_0 b_0 = L_0 d_0 = 0 \). Thus \( b_0 \) and \( d_0 \) are proportional to \( \phi \). Balancing the \( x^{\sigma} \) terms and \( x^{\sigma} \ln x \) in equations (A3) then gives

\[
\begin{align*}
(\sigma + 1)(L_0 b_1 + L_1 d_1) + (L_0 d_1 + L_1 b_1) &= 0, \\
(\sigma + 1)(L_0 b_0 + L_1 d_0) &= 0,
\end{align*}
\]

which (since we are supposing that \( \sigma 
eq -1 \)) implies

\[
\begin{align*}
L_0 b_1 + L_1 b_0 &= 0, \\
L_0 d_1 + L_1 d_0 &= 0.
\end{align*}
\]

The solvability condition (SC) then implies that \( b_0 = d_0 = 0 \), which contradicts the assumption that at least one of \( a_0, b_0, c_0 \) and \( d_0 \) must be nonzero. Therefore, in the resonant case, \( \sigma \) must take the value 0 or -1.

As will become evident, \( \sigma = 0 \) and \( \sigma = -1 \) provide the regular and singular solutions respectively.

**Absence of Logarithmic Terms in the Regular (\( \sigma = 0 \)) Solution**

Consider \( \sigma = 0 \). Balancing the \( x^{n-1} \ln x \) terms in equations (A3) yields

\[
\begin{align*}
i\lambda B^2 n c_n &= -(L_0 d_{n-1} + \ldots + L_0 d_0) \\
&\quad + \lambda^2 B^2 d_{n-1} \\
nL_0 d_n &= -n(L_0 c_{n-1} + \ldots + L_0 c_0) \\
&\quad + i\lambda(L_0 c_{n-1} + \ldots + L_0 c_0)
\end{align*}
\]

for \( n \geq 1 \). Moreover, from equations (A5) \( c_0 = L_0 d_0 = 0 \). We now prove by induction that the functions \( c_n, d_n \) are all identically zero.

**Proof.** Suppose that \( c_n = 0 \) for all \( n \leq N \), that \( d_n = 0 \) for all \( n \leq N - 1 \) and that \( L_0 d_N = 0 \). (This is certainly the case for \( N = 0 \).) Then we shall show that \( c_{N+1} = 0 \), that \( d_N = 0 \) and that \( L_0 d_{N+1} = 0 \). Setting \( n = N + 1 \) in equation (A8a) yields \( c_{N+1} = 0 \). In the nonresonant case \( d_N = 0 \) follows trivially from \( L_0 d_N = 0 \); and setting \( n = N + 1 \) in equation (A8b) yields \( L_0 d_{N+1} = 0 \). In the resonant case, setting \( n = N + 1 \) in equation (A8b) gives, for the regular solution (\( \sigma = 0 \)),

\[
L_0 d_{N+1} + L_1 d_N = 0;
\]

hence from the solvability condition \( d_N = 0 \) and \( L_0 d_{N+1} = 0 \), as required. Thus by mathematical induction all the \( c_n \) and \( d_n \) are zero, if \( \sigma = 0 \). QED

This justifies our use of the term “regular” to describe \( \sigma = 0 \) solutions.

**Solutions in Nonresonant Case: Lowest Order Terms**

When \( \sigma = 0 \), (so that \( c_0 = d_0 = 0 \)) balancing the \( x^0 \) terms in equations (A3) yields

\[
\begin{align*}
-\lambda B^2 a_1 &= (L_0 - \lambda^2 B^2)b_0, \\
L_0(b_1 - i\lambda a_0) &= -L_1 b_0,
\end{align*}
\]

and balancing the \( x^1 \) terms yields

\[
\begin{align*}
-2\lambda B^2 a_2 &= (L_0 - \lambda^2 B^2)b_1 + L_1 b_0 \\
&= -\lambda^2 B^2 b_1 + i\lambda L_0 a_0, \\
L_0(2b_2 - i\lambda a_1) &= -2(L_1 b_1 + L_2 b_0) + i\lambda L_1 a_0.
\end{align*}
\]

In the nonresonant case, \( a_0 \) and \( b_0 \) are arbitrary functions of \( z \). Since the governing equations are linear, we may consider the two solutions \( a_0 \neq 0, b_0 = 0 \) and \( b_0 \neq 0, a_0 = 0 \) separately: the general solution is a linear combination of these two.

**Solution with \( b_0 = 0 \).** Since \( L_0 \) is invertible in the nonresonant case, equations (A9a) and (A9b) give

\[
\begin{align*}
a_1 &= 0, \\
b_1 &= i\lambda a_0,
\end{align*}
\]

and equations (A9c) and (A9d) give

\[
\begin{align*}
a_2 &= \frac{-1}{2\lambda B^2}(L_0 - \lambda^2 B^2)a_0, \\
b_2 &= \frac{-i\lambda}{2}L_0^{-1}(L_1 a_0).
\end{align*}
\]

**Solution with \( a_0 = 0 \).** Equations (A9a) and (A9b) give

\[
\begin{align*}
a_1 &= \frac{-1}{i\lambda B^2}(L_0 - \lambda^2 B^2)b_0, \\
b_1 &= -L_0^{-1}(L_1 b_0),
\end{align*}
\]

and it follows from equations (A9c) and (A9d) that

\[
\begin{align*}
a_2 &= \frac{1}{2}i\lambda L_0^{-1}(L_1 b_0), \\
b_2 &= L_0^{-1}(L_0 L_0^{-1}L_1 b_0 - L_0 b_0) - \frac{1}{2\lambda B^2}(L_0 - \lambda^2 B^2)b_0.
\end{align*}
\]

**Regular (\( \sigma = 0 \)) Solutions in Resonant Case: Lowest Order Terms**

In this case the equations (A9a)-(A9d) still apply. We supplement these with two further equations, obtained by balancing \( x^2 \) terms in equations (A3):

\[
\begin{align*}
-3i\lambda B^2 a_3 &= (L_0 - \lambda^2 B^2)b_2 + L_1 b_1 + L_2 b_0 \\
&= -\lambda^2 B^2 b_2 + \frac{1}{2}i\lambda(L_0 a_1 + L_1 a_0), \\
L_0(3b_3 - i\lambda a_2) &= -3(L_1 b_2 + L_2 b_1 + L_3 b_0) + i\lambda(L_1 a_1 + L_2 a_0).
\end{align*}
\]

Once again the function \( a_0 \) is arbitrary. From (A9b) and the solvability condition, \( b_0 \) must satisfy \( \langle \phi, L_1 \phi \rangle = 0 \); but otherwise it is arbitrary. As in the nonresonant case, we can consider the cases \( a_0 = 0 \) and \( b_0 = 0 \) separately. By linearity, the general regular solution is a linear combination of these two solutions.
Solution with $b_0 = 0$. It follows from equations (A9a) and (A9b) that
\[ a_1 = 0, \quad b_1 = i\lambda a_0 + \beta_1 \phi_r, \]
where $\beta_1$ is a constant that has yet to be determined. From equation (A9d) and the solvability condition we deduce that
\[ 0 = \langle \phi_r, L_1 (2b_1 - i\lambda a_0) \rangle = 2\beta_1 \langle \phi_r, L_1 \phi_r \rangle + i\lambda \langle \phi_r, L_1 a_0 \rangle, \]
whence
\[ \beta_1 = \frac{-i\lambda \langle \phi_r, L_1 a_0 \rangle}{2\langle \phi_r, L_1 \phi_r \rangle}. \]
Equation (A9c) gives
\[ a_2 = -\frac{1}{2B^2}(C_0 - \lambda^2 B^2)a_0 - \frac{1}{2}i\lambda \beta_1 \phi_r, \]
and equation (A9d) yields
\[ b_2 = -\frac{1}{2}L_0^{-1}(L_1(i\lambda a_0 + 2\beta_1 \phi_r)) + \beta_2 \phi_r. \]
Note that $\beta_1$ was defined in a manner such that the inverse $L_0^{-1}$ is well-defined in this equation. (The inverse operator is defined by equations (14) and (15).) The constant $\beta_2$ follows from equation (A9f) and the solvability condition:
\[ \beta_2 = \left\{ -2\langle \phi_r, L_2 (2i\lambda a_0 + 3\beta_1 \phi_r) \rangle + 3\langle \phi_r, L_1 L_0^{-1} L_1 a_0 \rangle \right\}/6\langle \phi_r, L_1 \phi_r \rangle. \]

Solution with $a_0 = 0$. We recall that $b_0$ is arbitrary except for the single restriction
\[ \langle \phi_r, L_1 b_0 \rangle = 0. \]
Equations (A9a) and (A9b) imply that
\[ a_1 = -\frac{1}{i\lambda B^2}(C_0 - \lambda^2 B^2)b_0, \quad b_1 = -L_0^{-1}(L_1 b_0) + \gamma_1 \phi_r, \]
where the constant $\gamma_1$ is determined from applying the solvability condition to equation (A9d):
\[ \gamma_1 = \frac{\langle \phi_r, L_1 L_0^{-1} L_1 b_0 \rangle - \langle \phi_r, L_2 b_0 \rangle}{\langle \phi_r, L_1 \phi_r \rangle}. \]

Singular ($\sigma = -1$) Solution: Lowest Order Terms

In the case $\sigma = -1$, equations (A5) give
\[ a_0 = 0, \quad b_0 = \beta_0 \phi_r, \quad c_0 = 0, \quad d_0 = \delta_0 \phi_r, \quad (A10) \]
where $\beta_0$ and $\delta_0$ are constants. Equation (A6a) still holds; and hence by the solvability condition (noting that $\sigma + 1 = 0$ and once again assuming $\langle \phi_r, L_1 \phi_r \rangle \neq 0$, $\delta_0 = 0$ and so $d_0 = 0$. (One can also get this from the $x^0 \ln x$ term in equation [A3a], without invoking the SC.) The constant $\beta_0$ is undetermined.

The only remaining nontrivial equations that come from balancing $x^0$ terms and $x^0 \ln x$ terms in equations (A3) are
\[ -i\lambda B^2 c_1 = -\lambda^2 B^2 b_0, \quad L_0 d_1 = 0; \quad (A11) \]
whence
\[ c_1 = -i\lambda \beta_0 \phi_r, \quad d_1 = \delta_1 \phi_r \quad (A12) \]
(with $\delta_1$ still to be determined). The function $a_1$ is undetermined. But $a_1$ in the singular solution is the coefficient of $x^0$ in $c_1$, as is $a_0$ in the regular solution. Since $a_0$ in the regular solution is quite arbitrary, we may without loss of generality set $a_1$ here to be zero; this corresponds to adding a chosen amount of regular solution to the singular solution.

Going to the next order we obtain
\[ -i\lambda B^2(a_2 + c_2) = (L_0 - \lambda^2 B^2)b_1 + \beta_0 L_2 \phi_r \quad (A13a) \]
\[ -i\lambda B^2 c_2 = -\lambda^2 B^2(\delta_2 \phi_r) \quad (A13b) \]
\[ L_0 d_1 + L_1 \delta_1 \phi_r = 0 \quad (A13c) \]
\[ L_0 b_2 + L_1 b_1 + \beta_0 L_2 \phi_r = 0. \quad (A13d) \]
The SC and (A13c) imply that $\delta_1 = 0$; so $d_1 = 0$. Thus (A13b) and (A13c) yield $c_2 = 0$ and $d_2 = \delta_2 \phi_r$. The SC and (A13d) give
\[ \langle \phi_r, L_1 b_1 \rangle = -\beta_0 \langle \phi_r, L_2 \phi_r \rangle. \quad (A14) \]

It is convenient to write $b_1$ in terms of two contributions, one of which is proportional to $\phi_r$
\[ b_1 = \beta_1 \phi_r + \Delta b_1 \quad (A15) \]
where
\[ \beta_1 = -\beta_0 \langle \phi_r, L_2 \phi_r \rangle \quad (A16) \]
and $\Delta b_1$ is arbitrary except that it must satisfy $\langle \phi_r, L_1 \Delta b_1 \rangle = 0$. Note that this is precisely the condition that is satisfied by the otherwise arbitrary $b_0$ of the regular solution (see above), so choosing a different $\Delta b_1$ simply means adding a different amount of the regular solution onto the singular solution. Note also that $a_0$ and $b_0$ in the regular solution are independent, so that we may independently choose $a_1$ and $\Delta b_1$ in the singular solution to be whatever we like. We shall generally take $\Delta b_1$ to be zero unless otherwise indicated.

Finally at this order, (A13a) and (A13d) imply that
\[ a_2 = -\frac{1}{i\lambda B^2} \{ (C_0 - \lambda^2 B^2)b_1 + \beta_0 L_2 \phi_r \} \quad (A17a) \]
\[ b_2 = -L_0^{-1}(L_1 b_1 + \beta_0 L_2 \phi_r) + \beta_2 \phi_r. \quad (A17b) \]

At the next order we find
\[ -i\lambda B^2(2a_3 + c_3) = (L_0 - \lambda^2 B^2)b_2 + \beta_1 \phi_r \quad (A18a) \]
\[ -2i\lambda B^2 c_3 = -\lambda^2 B^2(\delta_2 \phi_r) \quad (A18b) \]
\[ 2(L_0 d_2 + L_1 (\delta_2 \phi_r)) = \lambda^2 \beta_0 L_1 \phi_r \quad (A18c) \]
\[ 2(L_0 b_3 + L_1 b_2 + \beta_0 L_3 \phi_r) \quad (A18d) \]
- we have used the fact that $c_1 = -i\lambda \beta_0 \phi_r$. From the SC and (A18c) one finds that $\delta_2 = \frac{1}{2} \lambda^2 \beta_1$; thus $d_2$ and $c_3$ are determined. Moreover, the SC and (A18d) determines $\beta_2$ and hence $b_2$:
\[ b_2 = -\frac{1}{4} \lambda^2 \beta_0 + \left\{ \langle \phi_r, L_1 L_0^{-1} [(\beta_1 L_1 + \beta_0 L_2) \phi_r] \rangle \right\}/\langle \phi_r, L_1 \phi_r \rangle. \quad (A19) \]
APPENDIX B: RECURSION FORMULAE FOR THE COEFFICIENTS OF THE GENERALIZED FROBENIUS SOLUTIONS

In this appendix it is demonstrated explicitly that the coefficients in each generalized Frobenius series (i.e., for the nonresonant case and for both regular and singular solutions in the resonant case) are expressible in terms of lower-order coefficients in the series, so that finding the series is a well-defined procedure once the first few terms are known.

Nonresonant Case

In this case \( \sigma = 0 \) and the \( c_n \) and \( d_n \) are all identically zero (see Appendix A). It then follows from equations (A3) that for \( n \geq 0 \)

\[
-\lambda B^2 (n+1)a_{n+1} = \sum_{m=0}^{n} L_{n-m} b_m - \lambda^2 B^2 b_n \quad (B1)
\]

and

\[
(n+1) \sum_{m=0}^{n+1} L_{n-m+1} b_m = i\lambda \sum_{m=0}^{n} L_{n-m} a_m . \quad (B2)
\]

Since \( L_0 \) is invertible in the nonresonant case, equation (B2) can be rewritten

\[
b_{n+1} = -L_0^{-1} \sum_{m=0}^{n} L_{n-m+1} b_m + \frac{i\lambda}{n+1} L_0^{-1} \sum_{m=0}^{n} L_{n-m} a_m . \quad (B3)
\]

It is readily apparent from equations (B1) and (B3) that \( a_{n+1} \) and \( b_{n+1} \) are determined by the previous \( a_n \)'s and \( b_m \)'s: thus all the coefficients are determined once \( a_0 \) and \( b_0 \) (which are arbitrary) have been chosen.

Resonant Case: Regular Solution

As in the nonresonant case, \( \sigma = 0 \), the \( c_n \) and \( d_n \) are all identically zero and the coefficients satisfy equations (B1) and (B2). The difference now is that \( L_{-1}^{-1} \) is only defined if \( \phi g = 0 \) (the solvability condition SC) and even then it is only defined up to an arbitrary additive multiple of \( \phi \). Clearly equation (B1) determines \( a_{n+1} \) in terms of \( b_0, \ldots, b_n \). It remains to be shown that \( b_{n+1} \) is determined by \( a_0, \ldots, a_n \) and \( b_0, \ldots, b_n \). Consider then equation (B2) for \( n = 0 \):

\[
L_0 b_1 + L_1 b_0 = i\lambda L_0 a_0 . \quad (B4)
\]

Using the solvability condition this implies that \( b_0 \) must satisfy \( (\phi L_1 b_0) = 0 \) (but in fact is otherwise arbitrary) and that \( b_1 = i\lambda a_0 - L_0^{-1} L_1 b_0 + \beta_1 \phi \), where \( \beta_1 \) has yet to be determined. We now proceed to prove that \( b_{n+1} \) is determined by the previous \( a_n \)'s and \( b_m \)'s using mathematical induction.

Proof. Suppose that \( a_0, \ldots, a_{N-1} \) and \( b_0, \ldots, b_{N-1} \) are already determined and that \( b_N \) is also determined except for the constant \( \beta_N \):

\[
b_N = b_N(a_0, \ldots, a_{N-1}, b_0, \ldots, b_{N-1}) + \beta_N \phi . \quad (B5)
\]

Note that we know this to be true for \( N = 1 \), once \( a_0 \) and \( b_0 \) have been chosen. We shall demonstrate that \( \beta_N \) can be determined in terms of the \( a_m \)'s and \( b_m \)'s for \( m < N \) and that equation (B5) with \( N \) replaced by \( N + 1 \) then follows. The constant \( \beta_N \) is obtained from equation (B2) with \( n = N \); it is useful first to rewrite this as

\[
L_0(b_{N+1} - i\lambda a_N + \beta_N L_1 \phi_1 = -L_1 b_N - \sum_{m=0}^{N-1} L_{N-m+1} b_m + \frac{i\lambda}{N+1} \sum_{m=0}^{N-1} L_{N-m} a_m . \quad (B6)
\]

Applying the SC to this determines \( \beta_N \), in terms of the \( a_m \) and \( b_m \) for \( m < N \) only. Note that \( \beta_N \) doesn't depend on \( a_N \). Thus \( b_N \) is completely determined; and \( a_N \) can also be found from the previous \( b \)'s. Finally, having determined \( \beta_N \) to satisfy the SC, equation (B6) can be solved to find \( b_{N+1} \) in terms of \( a_m \), \( b_m \) (\( m = 0, \ldots, N \)) and an as yet undetermined constant \( \beta_{N+1} \), in the form of equation (B5). This completes the proof by induction. QED

Resonant Case: Singular Solution

In this case \( \sigma = -1 \), and it follows from equations (A3), balancing both \( x^{\sigma+n} \) terms and \( x^{\sigma+n} \ln x \) terms that for \( n \geq 1 \)

\[
-i\lambda B^2 (n a_{n+1} + c_{n+1}) = -\lambda^2 B^2 b_n + \sum_{m=0}^{n} L_{n-m} b_m \quad (B7a)
\]

\[
-i\lambda B^2 n c_{n+1} = -\lambda^2 B^2 d_n + \sum_{m=0}^{n} L_{n-m} d_m \quad (B7b)
\]

\[
\sum_{m=0}^{n+1} L_{n-m+1} b_m = i\lambda \sum_{m=0}^{n} L_{n-m} a_m \quad (B7c)
\]

\[
\sum_{m=0}^{n+1} L_{n-m+1} d_m = i\lambda \sum_{m=0}^{n} L_{n-m} c_m . \quad (B7d)
\]

These may be re-arranged in the more convenient form

\[
-i\lambda B^2 n^2 a_{n+1} = -\lambda^2 B^2 (n b_n - d_n) + \sum_{m=0}^{n} L_{n-m} (n b_m - d_m) \quad (B8a)
\]

\[
-i\lambda B^2 n c_{n+1} = -\lambda^2 B^2 d_n + \sum_{m=0}^{n} L_{n-m} d_m (B8b)
\]

\[
n^2 \sum_{m=0}^{n+1} L_{n-m+1} b_m = i\lambda \sum_{m=0}^{n} L_{n-m} (n a_m - c_m) \quad (B8c)
\]

\[
n \sum_{m=0}^{n+1} L_{n-m+1} d_m = i\lambda \sum_{m=0}^{n} L_{n-m} c_m . \quad (B8d)
\]

In addition, it follows (see Appendix A) from balancing the lower-order terms in equations (A3) that

\[
a_0 = c_0 = d_0 = 0, b_0 = \beta_0 \phi_1, c_1 = -i\lambda b_0, d_1 = \delta_1 \phi_1 \quad (B9)
\]

with \( \beta_0 \) and \( a_1 \) arbitrary. Also (which comes from applying the SC to (B8c))

\[
b_1 = \frac{\phi_1 L_2 b_0}{\phi_1 L_1 \phi_1} + \Delta b_1 \quad (B10)
\]
(Appendix A), where \( \Delta b \) satisfies \( \langle \phi, \mathcal{L}_1 \Delta b \rangle \) but is otherwise arbitrary. Equation (B8c) can now be solved for \( b_2 \), except that \( b_2 \) contains an (as yet) undetermined additive multiple of the resonant eigenfunction. It is clear from (B8a) and (B8b) that \( a_{n+1}, c_{n+1} (n \geq 1) \) are determined once \( b_m \) and \( d_m (m \leq n) \) are known. Now equation (B8d) is essentially the same as equation (B2), with \( a_m \) and \( b_m \) replaced by \( c_m \) and \( d_m \); furthermore, we know \( a_0, b_0 \) and \( c_0 \) and \( d_0 \) and \( d_1 \) is known to within an additive multiple of \( \phi_r \). Hence the induction proof of the previous section shows that \( d_{n+1} (n \geq 0) \) is determined once the \( c_m (m \leq n) \) are known. The same induction argument can be applied to equation (B8c) for the case of \( b_{n+1} \), as \( a_0, \ldots, d_0, a_1, \ldots, d_1 \) can now be taken as known and \( b_2 \) is of the required form of being known except for an additive multiple of \( \phi_r \).

Acknowledgments. We thank the referees for constructive comments. This work is supported by the UK Science and Engineering Research Council.

The Editor thanks L. Chen and M. E. Engebretson for their assistance in evaluating this manuscript.

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M. J. Thompson, Astronomy Unit, School of Mathematical Sciences, Queen Mary and Westfield College, London E1 4NS England.

A. N. Wright, Department of Mathematical and Computational Sciences, University of St. Andrews, Fife KY16 9SS Scotland.

(Received November 10, 1992; revised February 11, 1993; accepted March 19, 1993.)