Abstract

The local configurations of three-dimensional magnetic neutral points are investigated by a linear analysis about the null. It is found that the number of free parameters determining the arrangement of field lines is four. The configurations are first classified as either potential or non-potential. Then the non-potential cases are subdivided into three cases depending on whether the component of current parallel to the spine is less than, equal to or greater than a threshold current (the square root of the discriminant of the characteristic equation of the symmetric part of the matrix $M$). Therefore there are three types of linear non-potential null configurations (a radial null, a critical spiral and a spiral). The effect of the four free parameters on the system is examined and it is found that only one parameter categorises the potential configurations, whilst two parameters are required if current is parallel to the spine. However, all four parameters are needed if there is current both parallel and perpendicular to the spine axis. The magnitude of the current parallel to the spine determines whether the null has spiral, critical spiral or radial field lines whilst the current perpendicular to the spine affects the inclination of the fan plane to the spine. A simple method is given to determine the basic structure of a null given $M$ the matrix which describes the local linear structure about a null point.
1 Introduction

Magnetic reconnection plays a central role in many phenomena that occur in plasmas. For example, in space, in the formation of x-ray bright points and solar flares on the Sun and in the interaction between the earth’s magnetosphere and the solar wind and, in the laboratory, in spheromaks. Over the last twenty years many aspects of two-dimensional reconnection have been extensively studied. In two dimensions the magnetic field vanishes at a neutral point which may be either ‘X’ type or ‘O’ type. In three dimensions papers\(^1,2\) have considered some aspects of magnetic reconnection at three-dimensional neutral points. We, in this paper, study such neutral points in detail by considering the local magnetic configurations that can occur around them.

To find the local magnetic structure about a neutral point we must consider the magnetic field in the neighbourhood of a point where the field vanishes (\(B\equiv0\)). If, without loss of generality, we take the neutral point to be situated at the origin and, in addition, assume that the magnetic field approaches zero linearly, the magnetic field \(B\) near a neutral point may be expressed to lowest order as

\[
B = M \cdot r,
\]

where \(M\) is a matrix with elements \(M_{ij} = \partial B_i/\partial x_j\) and \(r\) is the position vector \((x, y, z)^T\). In this paper we systematically study the matrix \(M\), firstly in two dimensions (Section 2) as a preliminary to the three-dimensional work of Section 3 where the matrix \(M\) is reduced to its simplest three-dimensional form and the theory used in calculating the magnetic configurations is discussed. Sections 4 and 5 discuss the potential and non-potential configurations, respectively. Finally in Section 6 this work is concluded.

2 Review of Two-Dimensional Neutral Points

In two dimensions the matrix \(M\) is simply

\[
M = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix},
\]

where \(a_{ij}\) are real constants. The solenoidal constraint \(\nabla \cdot B = 0\) gives \(a_{11} = -a_{22}\); thus the trace of \(M\) is zero. The diagonal entries in the matrix are associated with the potential part of the field so we let \(a_{11} = p\) and since the current associated with the neutral point is \(J = \mu_0 (0, 0, a_{21} - a_{12})\), we define,

\[
a_{12} = \frac{1}{2} (q - j_z) \quad \text{and} \quad a_{21} = \frac{1}{2} (q + j_z).
\]

Clearly, for a current-free neutral point \(a_{21} = a_{12} = q/2\) and the parameter \(q\) is therefore also associated with the potential field whilst \(j_z\) is the magnitude of the current perpendicular to the plane of the null point.

The matrix \(M\) may now finally be written as

\[
M = \begin{bmatrix}
p & \frac{1}{2} (q - j_z) \\
\frac{1}{2} (q + j_z) & -p
\end{bmatrix}.
\]

We will find it useful to define a threshold current,

\[
j_{\text{thresh}} = \sqrt{4p^2 + q^2},
\]

which we note only depends on the parameters associated with the potential part of the field. It is equal to the square root of the discriminant of the characteristic equation of the symmetric part of \(M\). We now calculate the flux function \(A\), which satisfies,

\[
B_X = \frac{\partial A}{\partial Y} \quad \text{and} \quad B_Y = -\frac{\partial A}{\partial X},
\]

so that

\[
A = \frac{1}{4} [(q - j_z) Y^2 - (q + j_z) X^2] + pXY.
\]

If we rotate the \(XY\)-axes through an angle \(\theta\) to give \(xy\)-axes using the relations

\[
\begin{aligned}
X &= x \cos \theta - y \sin \theta \\
Y &= x \sin \theta + y \cos \theta
\end{aligned}
\]

we find the local magnetic structure about a neutral point.
and substitute (4) into (3) with

\[ \tan 2\theta = -2 \frac{p}{q}, \]

with \( j_{\text{thresh}} \) as in (2) then \( A \) becomes

\[ A = \frac{1}{4} \left( (j_{\text{thresh}} - j_z) y^2 - (j_{\text{thresh}} + j_z) x^2 \right). \]

We thus see that in two dimensions the two parameters \( j_{\text{thresh}} \) and \( j_z \) govern the magnetic configuration.

The eigenvalues of the matrix \( M \) are given by

\[ \lambda = \pm \frac{1}{2} \sqrt{j_{\text{thresh}}^2 - j_z^2}; \]

therefore, depending on whether the current \( j_z \) is greater or less than the threshold value \( j_{\text{thresh}} \) the eigenvalues will be real or imaginary and the field will have a different structure.

In the following subsections a general two-dimensional null is studied firstly depending on whether it is potential (Section 2.1) or not (Section 2.2) and then whether the current is greater or less than \( j_{\text{thresh}} \) (Figure 1).

2.1 Potential Two-Dimensional Neutral Points

In the case of a current-free two-dimensional null \( j_z = 0 \), \( M \) is symmetric, and the eigenvalues are given by

\[ \lambda = \pm \frac{j_{\text{thresh}}}{2}; \]

therefore we have two real non-zero eigenvalues and consequently Equation (5) becomes

\[ A = \frac{j_{\text{thresh}}}{4} \left( y^2 - x^2 \right). \]

The field lines are therefore rectangular hyperbola and the separatrices intersect at an angle of \( \pi/2 \). This is an \( X \)-type neutral point, as shown in Figure 2a, and is the only possible configuration for a current-free two-dimensional neutral point.

2.2 Non-Potential Two-Dimensional Neutral Points

Two-dimensional neutral points with current are classified with respect to the magnitudes of \( j_z \) and \( j_{\text{thresh}} \).

2.2.1 \( |j_z| < j_{\text{thresh}} \)

When \( |j_z| < j_{\text{thresh}} \) the eigenvalues are real, equal in magnitude, but opposite in sign (det\( M < 0 \)). From the flux function we see that the field lines are hyperbolae with separatrices that intersect at an angle of

\[ \tan^{-1} \left( \frac{\left( j_{\text{thresh}}^2 - j_z^2 \right) \frac{1}{2}}{j_z} \right). \]

The null point formed is therefore an X-type neutral point as shown in Figure 2b. As \( j_z \to 0 \) the hyperbolae tends to a rectangular hyperbolae, thus reducing to the potential case. As the current increases, the angle between the separatrices increases as they close up along the \( y \)-axis.

2.2.2 \( |j_z| = j_{\text{thresh}} \)

The eigenvalues in this case are equal (det\( M = 0 \)), and so from Equation (6) they must be equal to zero. The flux function depends on either just \( x^2 \) if \( j_{\text{thresh}} = j_z \) or just \( y^2 \) if \( j_{\text{thresh}} = -j_z \); thus the configuration contains anti-parallel field lines with a null line along the \( y \)-axis or \( x \)-axis, respectively (Figure 2c).

2.2.3 \( |j_z| > j_{\text{thresh}} \)

If \( |j_z| > j_{\text{thresh}} \) the eigenvalues become complex conjugates (det\( M > 0 \)). When \( j_{\text{thresh}} = 0 \), \( p = q = 0 \) and the field configuration has circular field lines centred around the origin, whereas if \( j_{\text{thresh}} \neq 0 \) then the field contains concentric ellipses (Figure 2d).

3 Theory of Three-Dimensional Neutral Points

Figure 3a shows a three-dimensional neutral point formed by a field due to four point sources, two positive and two negative. If we look closely at the local structure near this null (Figure 3b) we find that there is a set of field lines extending into the null point and forming a surface (the thin lines) which we call the fan, following the nomenclature of Priest and Titov. However, only two field lines leave the null point (the
thick lines) and they are called the spine. These are the two basic components that make up the skeleton of any neutral point in three dimensions. The fan is a surface made up of field lines which radiate out, or into the null point (this is the same as the \( \sum \) surface referred to by Cowley\(^4\), Greene\(^5\) and Lau and Finn\(^1\)). The spine is made up of two special field lines that are directed away from the null if the field lines in the fan are directed towards the null and vice-versa (these are equivalent to the \( \gamma \) line\(^4,5,1\)). Field lines that lie near the null point, but do not pass through it, form bundles around the spine which spread out either side of the fan surface.

Mathematically, the linearised field about a three-dimensional neutral point may be described using Equation (1) in terms of a 3\( \times \)3 matrix of the form,

\[
\mathbf{M} = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix},
\]

where \( a_{ij} \) are real constants. The constraint \( \nabla \cdot \mathbf{B} = 0 \) implies that the trace of \( \mathbf{M} \) must be zero giving

\[
a_{11} + a_{22} + a_{33} = 0.
\]

This condition also implies that the eigenvalues \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) associated with the matrix sum to zero. The eigenvectors associated with these eigenvalues are \( \mathbf{x}_1 \), \( \mathbf{x}_2 \), and \( \mathbf{x}_3 \).

If a magnetic field line near the null is written in terms of a position vector \( \mathbf{r} = (x, y, z)^T \) which is dependent on an arbitrary parameter \( k \) then we may write

\[
\frac{d\mathbf{r}(k)}{dk} = \mathbf{M} \mathbf{r}(k) = \mathbf{B},
\]

using the substitution \( \mathbf{r}(k) = \mathbf{P} \mathbf{u}(k) \), where \( \mathbf{P} \) is the matrix of the eigenvectors of \( \mathbf{M} \), Equation (8) becomes

\[
\frac{d\mathbf{u}}{dk} = \mathbf{P}^{-1} \mathbf{M} \mathbf{P} \mathbf{u}.
\]

There are now two cases we must consider depending on whether the matrix \( \mathbf{M} \) can or can not be diagonalised. Firstly, if \( \mathbf{M} \) is diagonalisable to a matrix \( \mathbf{A} \), say, which may have real or complex elements then the above equation may be simply solved to give

\[
\mathbf{u} = \mathbf{A} \exp(\mathbf{A}k),
\]

where \( \mathbf{A} \) is also a diagonal matrix with entries \( A, B \) and \( C \) which are constant along a field line and implies

\[
\mathbf{r}(k) = A e^{\lambda_1 k} \mathbf{x}_1 + B e^{\lambda_2 k} \mathbf{x}_2 + C e^{\lambda_3 k} \mathbf{x}_3.
\]

Thus each field line may be written in terms of the eigenvalues and eigenvectors of the matrix \( \mathbf{M} \).

We initially consider the situation where all the eigenvalues are real. Since they sum to zero there is always one eigenvalue of opposite sign to the other two, say for example \( \lambda_1, \lambda_2 > 0, \lambda_3 < 0 \). If we trace a field line backwards away from the neutral point, that is let \( k \to -\infty \) in Equation (10), we find

\[
\mathbf{r}(k) \to C e^{\lambda_3 k} \mathbf{x}_3,
\]

so all the field lines that head in towards the null are parallel to the single eigenvector \( \mathbf{x}_3 \). However, if we trace forward along field lines away from the null, then \( k \to \infty \) and

\[
\mathbf{r}(k) \to A e^{\lambda_1 k} \mathbf{x}_1 + B e^{\lambda_2 k} \mathbf{x}_2.
\]

This implies that the field lines that are directed away from the null lie parallel to the plane defined by the eigenvectors \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \). If we compare this with our geometrical understanding of a three-dimensional null then we find that the eigenvector \( \mathbf{x}_3 \) with negative eigenvalue \( \lambda_3 \) defines the path of the spine, whilst the plane of the fan is defined by the eigenvectors \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \). Note that we traced backwards along the field lines when \( \lambda_3 < 0 \) and that the magnetic field of the spine is therefore heading towards the null whereas the field lines in the fan emanate radially outwards from the null and were associated with positive eigenvalues.

In the situation where we have two complex and one real eigenvalue, say \( \eta \pm i\nu \) and \(-2\eta\), with corresponding eigenvectors \( \mathbf{x}_1 = (x'_1 + ix'_2)/2, \mathbf{x}_2 = (x'_1 - ix'_2)/2 \) and \( \mathbf{x}_3 \), respectively, then

\[
\mathbf{r}(k) = \frac{1}{2} (A + iB) e^{(\eta + i\nu)k} (x'_1 + ix'_2) + \frac{1}{2} (A - iB) e^{(\eta - i\nu)k} (x'_1 - ix'_2) + Ce^{-2\eta k} \mathbf{x}_3,
\]

where \( A, B \) and \( C \) are constant along a field line. This may be rewritten as

\[
\mathbf{r}(k) = e^{i\nu k} R \cos(\Theta + \nu) k x'_1
\]
\[ r(k) \rightarrow R e^{\eta k} \cos (\Theta + \nu) k x_1' + B e^{\lambda k} x_2^* + Ce^{-2\eta k} x_3, \]  

(11)

where \( A \) and \( B \) have been rewritten in terms of the constants \( R \) and \( \Theta \). So, if for example, we take \( \eta > 0 \), then as \( k \rightarrow \infty \) this equation reduces to

\[ r(k) \rightarrow Re^{\eta k} \cos (\Theta + \nu) k x_1' - Re^{\eta k} \sin (\Theta + \nu) k x_2'. \]

Thus the fan plane is defined by the vectors \( x_1' \) and \( x_2' \) and field lines in this plane will be spirals. The spine lies in the direction of the eigenvector \( x_3 \), since as \( k \rightarrow -\infty \)

\[ r(k) \rightarrow Ce^{-2\eta k} x_3. \]

The second case we must consider is when the matrix \( M \) is not diagonalisable. This occurs when two of the eigenvalues are repeated and the matrix can only reduces to a Jordan normal form \( (J_n) \) which looks like

\[
J_n = \begin{bmatrix} 
\lambda & 1 & 0 \\
0 & \lambda & 0 \\
0 & 0 & -2\lambda 
\end{bmatrix}.
\]

The equation for the field lines may then be written using the substitution \( r(k) = Pu(k) \), where this time \( P = (x_1, x_2, x_3) \) and \( x_1, x_2 \) and \( x_3 \) satisfy

\[
\begin{align*}
Mx_1 &= \lambda x_1 \\
Mx_2^* &= x_1 + \lambda x_2^* \\
Mx_3 &= -2\lambda x_3
\end{align*}
\]

such that

\[
\frac{du}{dk} = J_n u.
\]

The equation for a field line is therefore

\[ r(k) = (A + Bk) e^{\lambda k} x_1 + Be^{\lambda k} x_2^* + Ce^{-2\lambda k} x_3, \]

(13)

where \( A, B \) and \( C \) are all constant along a field line. Hence, if we assume \( \lambda > 0 \) then running forwards along a field line so that \( k \rightarrow \infty \) we find

\[ r(k) \rightarrow (A + Bk) e^{\lambda k} x_1 + Be^{\lambda k} x_2^*; \]

thus the field lines lie in planes parallel to \( x_1 \) and \( x_2^* \) whereas if we trace backwards along a field line then \( k \rightarrow -\infty \) and

\[ r(k) \rightarrow Ce^{-2\lambda k} x_3, \]

so that the field lines become parallel to the vector \( x_3 \). We therefore find that the spine is defined by the eigenvector relating to the single eigenvalue whereas the fan plane is defined by the single eigenvalue and the Jordan basis vector which are related to the repeated eigenvalue.

In general, therefore, the spine lies along the eigenvector of \( M \) that relates to the single eigenvalue whose sign is opposite to that of the real parts of the remaining eigenvalues. These remaining eigenvalues have vectors associated with them which define the fan plane and which depend on the nature of the eigenvalues as shown in Table 1.

It may easily be proved that if the real parts of two of the three eigenvalues are positive, say \( \lambda_1, \lambda_2 > 0 \), then the neutral point will have field lines in the fan directed away from the null and a spine pointing into the null along \( x_3 \). This type of null is called a positive neutral point. The determinant of the matrix \( M \) will always be negative for this type of null. However, if the real parts of two of the eigenvalues are negative, say \( \lambda_1, \lambda_2 < 0 \), then the fan plane will have field lines pointing into the null with the spine again lying along the eigenvector \( x_3 \), but this time directed away from the null. Not surprisingly this type of neutral point is called a negative neutral point and has determinant \( M \) greater than zero.

### 3.1 Reduction of M to its Simplest Form

In order to examine all possible configurations of the localised field about the neutral point we reduce \( M \) to the least number of free parameters. In doing so, it is important to remember that the matrix \( M \) determines all the physical characteristics of the field including its structure, current and associated Lorentz force. Thus, we do not consider the simplest mathematical form of \( M^0 \) but derive a form for the matrix that gives the simplest topological form for the null without loss of generality. First we note that the field always has at least one real eigenvalue whose sign will always be opposite to the real parts of the other two. We therefore choose
the local orthogonal coordinate system such that the
eigenvector corresponding to this eigenvalue is in the
z-direction, so that the spine is directed along the z-
axis. Additionally, the matrix may be further reduced
by rotating the \(xy\)-plane so that the new \(x\)-axis lies in
the direction of the resultant current in the \(xy\)-plane.
Finally, by dividing by a scaling factor the matrix re-
duces to,
\[
M = \begin{bmatrix}
1 & \frac{1}{2} (q + j_\parallel) & 0 \\
\frac{1}{2} (q - j_\parallel) & p & 0 \\
0 & j_\perp & -(p+1)
\end{bmatrix},
\]
where \(p \geq -1\) and \(q^2 \leq j_\parallel^2 + 4p\). The potential part of
the configuration is defined by the parameters \(p\) and \(q\)
with the current given by
\[
J = \frac{1}{\mu_0} (j_\perp, 0, j_\parallel),
\]
where \(j_\parallel\) is the component of current parallel to the
spine and \(j_\perp\) is the component of current perpendicular
to the spine. Another way of producing the form (14)
is by first splitting the matrix (7) into symmetric (\(S\))
and asymmetric (\(A\)) parts. Next diagonalise \(S\) so that
the axes are along the eigenvectors then rotate about
the \(z\)-axis to make the \(x\)-axis along \(j_\perp\). Finally rotate
about the \(y\)-axis so that the upper half \(y - z\) terms
in \(S\) and \(A\) cancel. Note that the usual reduction of
an arbitrary matrix to block diagonal form (i.e. (14)
with \(q = j_\perp = 0\)) does not allow all the possible field
configurations.

Finally, we shall define a threshold current \(j_{\text{thresh}}\)
which depends purely on \(p\) and \(q\) the potential field
parameters such that
\[
j_{\text{thresh}} = \sqrt{(p-1)^2 + q^2}.
\]
Again, as in the two-dimensional case, \(j_{\text{thresh}}^2\) is equal
to the discriminant of the characteristic equation of
the symmetric part (\(S\)) of \(M\). This implies that the
three eigenvalues \((\lambda_1, \lambda_2, \lambda_3)\) associated with \(M\) may
be written
\[
\begin{align*}
\lambda_1 &= \frac{p+1+\sqrt{j_{\text{thresh}}^2-j_\parallel^2}}{2} \\
\lambda_2 &= \frac{p+1-\sqrt{j_{\text{thresh}}^2-j_\parallel^2}}{2} \\
\lambda_3 &= -(p+1)
\end{align*}
\]
We see, similar to the two-dimensional case, that it
is the relative sizes of \(j_{\text{thresh}}\) and \(j_\parallel\) which determine
the nature of the eigenvalues and consequently the local
magnetic configuration about the null point.

Note, that in situations where \(j_\perp\) equals zero the
perpendicular component of current does not exist; there-
fore we have one further degree of freedom and can ro-
tate the matrix about the spine (\(z\)-axis) such that it
reduces to the form
\[
M = \begin{bmatrix}
1 & -\frac{1}{2}j_\parallel & 0 \\
\frac{1}{2}j_\parallel & p & 0 \\
0 & 0 & -(p+1)
\end{bmatrix}.
\]
thus, throughout this paper in studying configurations
where \(j_\perp = 0\) we assume \(q = 0\) without loss of generality.
Also note that having taken a scaling factor from
our matrix we are excluding the possibility of all the
elements of the trace equalling zero. This is of course
a special situation and in general does not arise; how-
ever for completeness we do mention such situations
when they arise. Further, it is worth mentioning that
since \(p \geq -1\) the eigenvalue relating to the spine is
always negative; hence all the three-dimensional con-
fugurations we consider in Sections 4 and 5 of this pa-
er are positive neutral points. This means that all the
field lines in the fan planes are emanating outwards and
the spines are composed of pairs of field lines directed
towards the origin.

We now briefly mention some of the previous work
on three-dimensional neutral points. Cowley\(^4\) studied
a current-free neutral point of the form
\[
B = (\alpha x, \beta y, -(\alpha + \beta)z),
\]
where \(\alpha\) and \(\beta\) are of the same sign, with eigenval-
es \(\alpha, \beta\) and \(- (\alpha + \beta)\). Cowley referred to the case
\(\alpha, \beta > 0\) as Type A, which we call a positive radial null.
Similarly, the case \(\alpha, \beta < 0\) was referred to as Type B
and is a negative radial null point. For \(\alpha = \beta\) field lines
from the neutral point radiate with equal spacing from
the null; thus we call this a proper radial null, whereas
for \(\alpha \neq \beta\) the field lines of the null are unevenly spaced
and are orientated in one preferential direction, known
here as the major axis of the fan; thus the null is called
an improper radial null.
Fukao et al.\textsuperscript{7} studied more general neutral points than Cowley\textsuperscript{4}. They considered a $3 \times 3$ matrix containing six parameters and found that when all three eigenvalues are real the null point is radial, but one real and two complex conjugate give field lines that spiral logarithmically in the fan plane — this type of null is known as a proper spiral null. They found that if there is no fan current the spine and fan are perpendicular. Two-dimensional neutral points were also found in special circumstances containing either a line of X-points or O-points depending on whether the eigenvalues are real or imaginary.

In this paper we extend the previous work undertaken on three-dimensional magnetic neutral point structure by studying comprehensively the most general form for the matrix $M$ which defines the local magnetic field about the null. With our matrix all linear magnetic field configurations that can arise are studied and at least two rotations and a multiplication by a scalar are needed simply to transform our set of linear null points into any other. Through our study, which follows a similar procedure to that undertaken in studying the two-dimensional neutral point in Section 2, we note that there are in fact extra configurations to those previously discovered. For instance, we find that logarithmically spiralling nulls are only one special case of the family of spiralling nulls and, in general, field lines in the fan of a spiral form a more complex spiralling pattern; these nulls are known as improper spiral nulls. Also we can explain the effects that the current has on the neutral points in determining their structure. Figure 4 illustrates how the three-dimensional null point structures may be divided with respect to the magnitudes of the components of current and indicates what types of null are studied in the following sections.

In the next two sections all the three-dimensional figures are illustrated as follows. The spine is plotted as a solid thick line in the $z$-direction. The fan plane is shown by the square region enclosed by dashed lines with the fan field lines themselves depicted by continuous lines. A bundle of field lines around the spine is illustrated by dashed lines and are drawn only below the fan plane for clarity.

### 4 Three-Dimensional Potential Nulls

The matrix $M$ representing the linear field about a potential three-dimensional null is symmetric and may be written as

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & -(p + 1) \end{bmatrix}.$$  

The eigenvalues relating to this matrix are

$$\lambda_1 = 1 \quad \lambda_2 = p \quad \text{and} \quad \lambda_3 = -(p + 1).$$

By our choice of the matrix $M$ we find we must have $p \geq 0$ such that the eigenvalue $-(p + 1)$ which corresponds to the eigenvector $-(p + 1)$ which forms the spine of the neutral point, as required. The eigenvectors $x_1, x_2$ and $x_3$ are

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

thus, as found by Fukao et al.\textsuperscript{7}, the fan plane is perpendicular to the spine in a potential situation.

The threshold current $j_{\text{thresh}} = |p - 1|$ in this situation. Depending on its value we have three cases to consider. Firstly, when $j_{\text{thresh}} = 0$ and $p > 0$ all the eigenvalues are non-zero, but two are equal (Section 4.1). Secondly, in Section 4.2 we examine the situation where all the eigenvalues are non-zero and unequal ($p > 0$, $j_{\text{thresh}} > 0$) and finally we have the case where one eigenvalue is zero, $p = 0$ (Section 4.3).

#### 4.1 $p > 0$, $j_{\text{thresh}} = 0$

Assuming $p > 0$ and $j_{\text{thresh}} = 0$ the only value that the parameter $p$ can take is $p = 1$ so that

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$  

We find now that two of the eigenvalues are repeated; however, three eigenvectors may still be found and are $i, j$ and $k$. The field associated with this matrix is
a positive proper radial null as depicted in Figure 5a. Here, since this is the first three-dimensional null illustrated in this paper, we also draw the field lines in the fan plane (xy-plane) (Figure 5b) and the field lines in the xz-plane which is perpendicular to the surface of the fan (Figure 5c).

4.2 \( p > 0, \ j_{\text{thresh}} > 0 \)

If \( p > 0 \) and \( j_{\text{thresh}} > 0 \) then improper radial nulls are formed with the field aligned predominantly in the direction of the eigenvector corresponding to the greatest eigenvalue of the two associated with the fan plane. The field lines rapidly curve such that they run parallel to the \( x \)-axis if \( 0 < p < 1 \) and parallel to the \( y \)-axis if \( p > 1 \) (see Figures 5d and 5e respectively). This is as predicted for, if we consider Equation (10) for a field line in this particular case, we find that as \( k \to \infty \)

\[
    r(k) \rightarrow A e^{k} i + B e^{pk} j .
\]

So if \( 0 < p < 1 \) then \( r(k_{\infty}) \approx A e^{k} i \) and the field lines lie in the \( xy \)-plane, but are inclined along the major fan axis \( y = 0 \), whereas if \( p > 1 \) then \( r(k_{\infty}) \approx B e^{pk} j \) and the major fan axis of the improper null is the \( x = 0 \) line.

4.3 \( p = 0 \)

When \( p = 0 \) the matrix \( M \) reduces to

\[
    M = \begin{bmatrix}
        1 & 0 & 0 \\
        0 & 0 & 0 \\
        0 & 0 & -1
    \end{bmatrix},
\]

thus the field reduces to the two-dimensional case with potential X-points lying in planes parallel to the \( xz \)-planes and forming a null-line along the \( y \)-axis (Figure 5f). Note that if a scaling factor had not been taken from the matrix the only extra possible field line configuration is the trivial situation of \( B = 0 \).

5 Three-Dimensional Non-Potential Nulls

Here the matrix \( M \) is asymmetric and has an associated current \( J = (j_{\perp}, 0, j_{\parallel}) \). The eigenvalues of the matrix \( M \) are

\[
    \lambda_{1,2} = \frac{1}{2} (p + 1) \pm \frac{1}{2} \sqrt{j_{\text{thresh}}^{2} - j_{\parallel}^{2}} \\
    \lambda_{3} = -(p + 1)
\]

where \( p \geq -1 \), \( (p + 1)^{2} \geq j_{\text{thresh}}^{2} - j_{\parallel}^{2} \) and \( j_{\text{thresh}}^{2} = (p - 1)^{2} + q^{2} \), as previously defined. These constraints are necessary to ensure that the eigenvalue \( \lambda_{3} \) always corresponds to the eigenvector that defines the spine of the null.

5.1 \( |j_{\parallel}| < j_{\text{thresh}} \)

First let us consider the situation where the magnitude of the component of current parallel to the spine is less than that of the threshold current. This implies that all three eigenvalues are real and distinct and all three eigenvectors exist.

5.1.1 \( j_{\perp} = 0 \) and \( j_{\parallel} \neq 0 \)

The perpendicular component of current is zero here so we may assume \( q = 0 \) and if \( p > -j_{\parallel}^{2}/4 \) the eigenvectors are found to be

\[
    x_{1,2} = \begin{pmatrix}
        1 - p \sqrt{j_{\text{thresh}}^{2} - j_{\parallel}^{2}} \\
        j_{\parallel} \\
        0
    \end{pmatrix}, \quad x_{3} = \begin{pmatrix}
        0 \\
        1 \\
        0
    \end{pmatrix},
\]

so the fan and spine are perpendicular (Figure 6a). Substituting \( \lambda_{i} \) and \( x_{i} \) into Equation (10) we find that as \( k \to \infty \) the field lines in the plane of the fan become parallel to the line

\[
    y = \frac{1 - p \sqrt{j_{\text{thresh}}^{2} - j_{\parallel}^{2}}}{j_{\parallel}} x .
\]

This may be formally shown by considering the equation for a field line in the plane of the fan which is,

\[
    r(k) = A e^{\lambda_{1} k} x_{1} + B e^{\lambda_{2} k} x_{2} .
\]
Since \( p > -j^2/4 \), \( \lambda_1 > \lambda_2 \) and so the major axis of the fan is defined by the eigenvector \( \mathbf{x}_1 \), and as \( k \to \infty \) the term \( Ae^{\lambda_1 k} \mathbf{x}_1 \) dominates. The field lines in the fan do not form the same sort of improper null as found in the potential situation but instead form a skewed improper null since the eigenvectors \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) are not perpendicular. When \( p = -j^2/4 \) the field reduces to a two-dimensional configuration (Figure 6b) containing collapsed X-points in planes parallel to the plane

\[
2px - j_\parallel y = 0,
\]

and a null line (dashed) along

\[
y = \frac{j_\parallel - \sqrt{4p - j^2}}{2} x.
\]

5.1.2 \( j_\perp \neq 0 \) and \( j_\parallel = 0 \)

If the current is purely perpendicular to the spine then \( j_\perp \neq 0 \) and \( j_\parallel = 0 \) and the eigenvalues for \( \mathbf{M} \) simply become

\[
\lambda_{1,2} = \frac{1}{2} (p + 1 \pm \sqrt{\lambda_{\text{thresh}}}), \quad \lambda_3 = -(p+1),
\]

where \( p > \lambda_{\text{thresh}} - 1 \). If \( p > \lambda_{\text{thresh}} - 1 \) the eigenvectors are given by

\[
\mathbf{x}_{1,2} = \begin{pmatrix}
\frac{-3p^2 + 3j_{\text{thresh}} + 2(p+2)j_{\text{thresh}}}{2j_\perp \sqrt{j_{\text{thresh}}^2 - (p-1)^2}} \\
\frac{3+3p+3j_{\text{thresh}}}{2j_\perp}
\end{pmatrix}
\]

\[
\mathbf{x}_3 = \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}.
\]

The plane of the fan is therefore not perpendicular to the spine (Figure 6c) and is defined by the equation

\[
2j_\perp \sqrt{j_{\text{thresh}}^2 - (p-1)^2} x - 4j_\perp (p+2) y + 
\left[9(p+1)^2 - j_{\text{thresh}}^2\right] z = 0.
\]

Note that, as \( j_\perp \) increases, the angle between the fan and spine reduces so that ultimately (\( j_\perp \to \infty \)) the spine lies in the fan plane. Also note that the fan does not necessarily tilt about the x-axis (the direction of the current) and so the perpendicular component of current does not in general lie in the plane of the fan. The field lines in the fan are positive improper nulls which orientate themselves predominately along the line

\[
1(\gamma) = \begin{pmatrix}
-3p^2 + 3j_{\text{thresh}} + 2(p+2)j_{\text{thresh}} \\
2j_\perp \sqrt{j_{\text{thresh}}^2 - (p-1)^2}
\end{pmatrix} \mathbf{\gamma},
\]

\[
+ \begin{pmatrix}
3+3p+3j_{\text{thresh}} \\
2j_\perp
\end{pmatrix} \mathbf{\gamma}, \mathbf{\gamma}, \mathbf{\gamma}, \mathbf{\gamma}
\]

where \( \mathbf{\gamma} \) is real. This line is associated with the eigenvalue \( \lambda_1 \) since \( \lambda_1 > \lambda_2 \). When \( p = \lambda_{\text{thresh}} - 1 \) the field reduces to a two-dimensional situation (Figure 6d) where successive X-points form in \( xz \)-planes with a null line along

\[
1(\gamma) = \begin{pmatrix}
-(p+1) \sqrt{p} \\
(p+1) \sqrt{p}
\end{pmatrix} \mathbf{\gamma}, \mathbf{\gamma}, \mathbf{\gamma}, \mathbf{\gamma}
\]

5.1.3 \( j_\perp \neq 0 \) and \( j_\parallel \neq 0 \)

When \( p > -1 \) and \( (p+1)^2 > j_{\text{thresh}}^2 - j_\parallel^2 \) and there are both parallel and perpendicular components of current the eigenvalues are as in Equation (18) and have corresponding eigenvectors

\[
\mathbf{x}_{1,2} = \begin{pmatrix}
\frac{-3p^2 + 3j_{\text{thresh}} + 2(p+2)j_{\text{thresh}}}{2j_\perp \sqrt{j_{\text{thresh}}^2 - (p-1)^2}} \\
\frac{3+3p+3j_{\text{thresh}}}{2j_\perp}
\end{pmatrix}
\]

\[
\mathbf{x}_3 = \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}.
\]

The fan is therefore tilted towards the spine and lies in the plane

\[
2j_\perp \left(\sqrt{j_{\text{thresh}}^2 - (p-1)^2} + j_\parallel\right) x - 4j_\perp (p+2) y + 
\left[9(p+1)^2 - j_{\text{thresh}}^2\right] z = 0.
\]

thus the current perpendicular to the spine does not lie in the plane of the fan. The field lines lying in the fan plane form a positive skewed improper null (Figure 6e) whose major fan axis is in the direction of the vector \( \mathbf{x}_1 \) since \( \lambda_1 > \lambda_2 \) and the term \( Ae^{\lambda_1 k} \mathbf{x}_1 \) dominates in the equation for a field line in the fan. The field reduces to a two-dimensional collapsed X-point when
\[ (p + 1)^2 > j_{\text{thresh}}^2 - j_\parallel^2 \] (Figure 6f). The plane of the X-point is
\[ 2px - \left( \sqrt{4p + j_\parallel^2} - j_\parallel \right) y = 0 , \]
and the null line lies along
\[ 1(\gamma) = \begin{pmatrix} \frac{p+1}{j_\perp} \gamma, \\ \gamma \end{pmatrix} . \]

5.2 \( |j_\parallel| = j_{\text{thresh}} \)

In the case where \(|j_\parallel| = j_{\text{thresh}}\) we find that two of the eigenvalues are repeated so that with \( p \geq -1 \),
\[ \lambda_{1,2} = \frac{p+1}{2} \quad \text{and} \quad \lambda_3 = -(p+1) . \]

5.2.1 \( j_\perp \neq 0 \) and \( j_\parallel = 0 \)

If the component of the current parallel to the spine is zero \((j_\parallel = j_{\text{thresh}} = 0)\) then we must have \( p = 1 \) and \( q = 0 \) since we require the \( z \)-axis to be the spine. The eigenvalues are therefore \( 1, 1, -2 \) and the corresponding eigenvectors are, respectively,
\[ \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ \frac{3}{j_\parallel} \\ 1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \]

Since the null is non-potential the component of current perpendicular to the spine is non-zero. Thus the fan does not lie in the \( xy \)-plane, but is in fact defined by the equation
\[ j_\perp y - 3z = 0 . \]

The field lines lying in the plane of the fan extend radially outwards and form what looks like a radial null (Figure 7a).

5.2.2 \( j_\perp = 0 \) and \( j_\parallel \neq 0 \)

When \( p \geq -1 \) and \(|j_\parallel| = j_{\text{thresh}} \neq 0\) we find that the two repeated eigenvalues have only one associated eigenvector, so to define the plane of the fan an extra vector must be calculated, known as a Jordan basis vector. This is found by solving
\[ \mathbf{M} \mathbf{x}_2 = \lambda \mathbf{x}_2 + \mathbf{x}_1 , \]
where \( \lambda \) is the repeated eigenvalue and \( \mathbf{x}_1 \) is its associated eigenvector (see Section 3).

If \( p > -1 \) \((p \neq 1)\) and the perpendicular current is zero we may assume \( q = 0 \) so the vectors which define the fan plane and spine of the null point are found to be
\[ \mathbf{x}_1 = \begin{pmatrix} 1-p \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 3-p \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ; \]
thus the fan plane is perpendicular to the spine. We find that the neutral point has a new form. It is neither an improper null nor, because of the straight lines in the fan, it is a spiral, and so we call it a critical spiral (Figure 7b). The predominant vector in the fan is the eigenvector \( \mathbf{x}_1 \) so the field lines are orientated towards this line as they move further from the null in the fan plane. This can be confirmed by considering the equation for a field line in the fan plane,
\[ \mathbf{r} = (A + Bk)e^{\lambda k} \mathbf{x}_1 + Be^{\lambda k} \mathbf{x}_2 , \]
whose dominant term is \( Be^{\lambda k} \mathbf{x}_1 \) when \( k \to \infty \).

If \( p = -1 \) then the neutral point merely reduces to a two-dimensional non-potential null with anti-parallel field lines such that the \( x = y : z \)-plane becomes a null plane (Figure 7c).

5.2.3 \( j_\perp \neq 0 \) and \( j_\parallel \neq 0 \)

When \( p > -1, j_\perp \neq 0 \) and the parallel component of the current is non-zero, but still equal to the threshold value, we again have repeated eigenvalues and have to look for a Jordan basis vector. The vectors are found to be
\[ \mathbf{x}_1 = \begin{pmatrix} \frac{3(p+1)(\sqrt{4p^2-(p-1)^2}-q_\parallel)}{2j_\perp(p-1)} \\ \frac{3(p+1)}{2j_\perp} \\ 1 \end{pmatrix} , \]
\[ \mathbf{x}_2 = \begin{pmatrix} \frac{-3p^2+4p+11}{2j_\perp+j_\parallel} \\ \frac{3p+5}{2j_\perp} \\ 1 \end{pmatrix} , \quad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \]
As in the previous cases we find that a critical spiral (Figure 7d) is created whose major fan axis is in the direction of the eigenvector $x_1$, but this time the fan lies in the plane

$$2 j_\perp \left( \sqrt{j_0^2 - (p-1)^2} + j_\parallel \right) x - 4 j_\perp (p+2) y + 9 (p+1)^2 z = 0 .$$

If $p = -1$ then parabolic field lines are formed lying in parallel $x=y+\text{constant};z$-planes (Figure 7e). The $x=y;z$-plane is a null plane and the parabolic field lines all have their turning points along the $y = 0$ line.

5.3 $| j_\parallel | > j_{\text{thresh}}$

When the parallel component of current is greater in magnitude than the threshold current two of the eigenvalues of $M$ will be complex conjugates,

$$\lambda_{1,2} = \frac{p+1}{2} \pm \frac{i}{2} \sqrt{j_0^2 - j_{\text{thresh}}^2} \quad \text{and} \quad \lambda_3 = -(p+1) .$$

Obviously the eigenvectors relating to the complex eigenvalues will also be complex conjugates; however we have already seen in Section 3 that if the complex vectors are split up into their real and imaginary parts then these resultant vectors define the plane of the fan.

5.3.1 $j_\perp = 0$ and $j_\parallel \neq 0$

Zero perpendicular current gives rise to a spine perpendicular to the fan consistent with the previous cases and has basis vectors

$$x'_1 = \begin{pmatrix} 1 \cr j_\parallel \cr 1 \end{pmatrix}, \quad x'_2 = \begin{pmatrix} \sqrt{j_0^2 - j_{\text{thresh}}^2} \cr j_\parallel \cr 0 \end{pmatrix},$$

$$x_3 = \begin{pmatrix} 0 \\
0 \\
1 \end{pmatrix},$$

where it is assumed that $q = 0$ and $p > -1$. The field lines in the fan plane form spirals of the form

$$\rho = \frac{C}{((p-1)\sin 2\phi + j_\parallel)^2} \times$$

$$\exp \left( \frac{(p+1)\tan^{-1} \left( \frac{j_\parallel \tan \phi + p-1}{(j_\parallel^2 - (p-1)^2)^{\frac{1}{2}}} \right)}{\left( j_\parallel^2 - (p-1)^2 \right)^{\frac{1}{2}}} \right) , \quad (19)$$

where $\rho = \sqrt{x^2 + y^2}$, $\phi = \tan^{-1} y/x$ and $C$ is an arbitrary constant. Remembering that $j_{\text{thresh}} = | p - 1 |$ in this case we find that no singularities arise as along as the condition $| j_\parallel | > j_{\text{thresh}}$ for a spiral holds. These are in general not logarithmic spirals (Figures 8a and 8c) contrary to$^{6,7}$. Logarithmic spirals only occur when $p = 1$ (Figure 8b) such that Equation (19) reduces to

$$\rho = D \exp \left( \frac{2\phi}{j_\parallel} \right) ,$$

where $D$ is an arbitrary constant. Note also that the associated vectors for the null are perpendicular if and only if $p = 1$. Some of the spirals are so weakly oscillating that they look more like improperly nulls (Figure 8a), whereas others are tightly coiled (Figure 8b). If we look at the equation for any general field line (not necessarily in the plane of the fan)

$$r(k) = e^{ik} R \cos (\Theta + \nu) k x'_1$$

$$- e^{ik} R \sin (\Theta + \nu) k x'_2 + C e^{-2\eta k} x_3 ,$$

we can easily see that field lines oscillates in the $x'_1$ and $x'_2$ directions, and so they spiral around the spine until they spread spiralling outwards parallel to the fan plane (Figure 8d).

When $p = -1$ then the field reduces to a two-dimensional null with a null line along the $z$-axis and elliptical field lines in successive $z=$constant-planes (Figure 8e). These elliptical field lines would become circular if the matrix had zero entries along the trace i.e. if we had not taken a scaling factor from the matrix.

5.3.2 $j_\perp \neq 0$ and $j_\parallel \neq 0$

It is not possible to create a spiral null without a component of current parallel to the spine, so we next consider $j f \neq 0$ and $j_\parallel \neq 0$. Basis vectors for this situation are found to be
ralling, as are the field lines around the spine. The field lines lying in the plane of the fan are spiralling the matrix from a mathematical viewpoint. This is the physical character of the field as opposed to reducing the 3-dimensional null we reduced the 3

comprehensively study the localised field about a three-dimensional configuration which are found to depend on four parameters. To particular values of \( p \) the null may reduce to a two-dimensional configuration containing successive X-points in parallel planes.

In a non-potential situation when the current is purely parallel to the spine of the null, just two parameters \( p \) and \( j_{||} \), the component of current parallel to the spine, are the minimum number of parameters required to define the character of the null. If the current is purely perpendicular to the spine three parameters \( p, j_{\text{thresh}} \) (or \( q \)) and \( j_{\perp} \) are needed to determine the null structure. However, if the current is inclined at neither 0° or 90° to the spine then four parameters \( p, j_{\text{thresh}} \) (or \( q \)), \( j_{||} \) and \( j_{\perp} \) are the minimum number required.

We find that the component of the current perpendicular to the spine determines the inclination of the fan plane with respect to the spine; note, however, that if \( j_{\perp} \) is non-zero then the other three parameters may vary the inclination of the plane too. Also the fan does not necessarily tilt about the \( x \)-axis, the direction of the perpendicular component of current, but in general it tilts about a different line in the \( xy \)-plane. The configuration of the field lines in the plane of the fan is determined mainly by the respective sizes of \( j_{||} \) and \( j_{\text{thresh}} \). However, the field lines may lie predominantly along one line known as the major axis of the fan; this is dependent on \( p, j_{\text{thresh}} \) and \( j_{\perp} \) as well as \( j_{||} \).

The only way of determining \( j_{||} \) and \( j_{\perp} \) is by calculating the eigenvalues, associated vectors and the current in the null. \( j_{\text{thresh}} \) on the other hand is easy to calculate since it is the discriminant of the characteristic equation of the symmetric part of the matrix \( M \). Knowing all this information will enable the exact structure of the null to be calculated and also the value of the parallel component of current that will deform your null from one type to another. There is, however, a

\[
\begin{align*}
x_1' &= \left( \frac{-3p^2+3j_{\text{thresh}}^2-j_x^2}{2j_x \left( \sqrt{j_{\text{thresh}}^2-(p-1)^2}+j_x \right)} \right), \\
x_2' &= \left( \frac{(p+2) \sqrt{j_{\text{thresh}}^2-j_x^2}}{j_x \left( \sqrt{j_{\text{thresh}}^2-(p-1)^2}+j_x \right)} \right), \\
x_3' &= \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right),
\end{align*}
\]

with \( p > -1 \). So, not surprisingly, the fan is not perpendicular to the spine (Figure 8c), but lies in the plane

\[
2j_x \left( \sqrt{j_{\text{thresh}}^2-(p-1)^2}+j_x \right) x - 4j_{\perp} (p+2) y + \left( 9 (p+1)^2 - j_{\text{thresh}}^2 + j_x^2 \right) z = 0.
\]

The field lines lying in the plane of the fan are spiralling, as are the field lines around the spine. When \( p = -1 \) the field reduces to a two-dimensional null with elliptical field lines in successive planes given by

\[
2j_x \left( \sqrt{j_{\text{thresh}}^2-4}+j_x \right) x - 4j_{\perp} y + \\
\left( j_x^2-j_{\text{thresh}}^2 \right) z = 0,
\]

which are inclined at an angle to the null line along the \( z \)-axis (Figure 8f).

6 Conclusion

In this paper we have analysed the local structure about a linear three-dimensional null. The type of field configurations are found to depend on four parameters. To comprehensively study the localised field about a three-dimensional null we reduced the 3×3 matrix which determined the field to its' simplest form by considering the physical character of the field as opposed to reducing the matrix from a mathematical view point. This enabled extra cases to be discovered which were overlooked by previous authors who tackled the problem from the mathematical angle.
relatively quick and easy way of discovering the basic structure of the null without solving the characteristic equation. That is by ascertaining whether $M$ is symmetric or not then by finding out what the sign of the determinant of $M$ and the sign of the discriminant of the characteristic equation of $M$ is. With these three facts you may determine whether the null is two- or three-dimensional, whether it is positive or negative, a spiral or improper radial null and also whether it is potential or non-potential (see Table 2 for details).

Finally, in our analysis we have considered the linearisation of the three-dimensional neutral point purely from the point of view of a cartesian geometry. Instead we could have linearised with respect to cylindrical or spherical coordinates, however, this approach does not lead to any new forms for the topology of the null. This is because discontinuities are introduced along $x = y = 0$ in both cylindrical and spherical polars in either the magnetic field or the current. These are unphysical since we have assumed our field and current are continuous in linearising about the null.

7 Acknowledgements

J.M. Smith wishes to thank EPSRC and C.E. Parnell, T. Neukirch and E.R. Priest wish to acknowledge PPARC for their financial support. The authors are also grateful to G.W. Inverarity for fruitful discussions and E.R. Priest is most grateful to T. Bogdon, B.C. Low and A. Hundhausen for their hospitality during his stay in Boulder.

8 References

**Figure Captions**

**Figure 1**: A categorisation of the different types of two-dimensional null and the respective limits of \( j_z \) (the \( z \)-component of current) and \( j_{\text{thresh}} \) (the threshold current) at which they occur.

**Figure 2**: Two-dimensional magnetic field plots of neutral points in the \( xy \)-plane showing (a) a potential X-point (\( j_z = 0 \)), (b) a non-potential X-point (\(| j_z | < j_{\text{thresh}}\)), (c) anti-parallel field lines (\(| j_z | = j_{\text{thresh}}\)) and (d) an elliptical null (\(| j_z | > j_{\text{thresh}}\)).

**Figure 3**: A three-dimensional potential configuration showing the global magnetic field structure due to four point sources (depicted by asterisks) containing two neutral points on the \( z=0 \)-plane\(^3\). (b) A schematic enlargement of one of the nulls showing the local structure about a three-dimensional negative neutral point with a fan (thin lines) and a spine (thick lines).

**Figure 4**: A categorisation of the various types of three-dimensional neutral point with respect to the relative sizes of \( j_\parallel \) (the component of current parallel to the spine of the null) and \( j_{\text{thresh}} \) (the threshold current).

**Figure 5**: The magnetic field configurations of three-dimensional potential fields. (a) The complete three-dimensional structure of a radial null (\( p = 1 \)) showing the field lines in (b) the fan plane (\( xy \)-plane) and (c) the \( xz \)-plane. An improper radial null with field lines aligned along (d) the \( x \)-axis and (e) the \( y \)-axis. (f) When \( p = 0 \) the null point reduces to a two-dimensional X-point field with the \( y \)-axis a null line (dashed).

**Figure 6**: Non-potential three-dimensional magnetic field configurations where the magnitude of the current parallel to the spine is less than the threshold current (\( q = 0 \)). (a) and (b) \( j_\perp = 0, j_\parallel \neq 0 \) and either \( p > -1 \) or \( p = -j_\parallel^2/4 \), respectively. (c) and (d) \( j_\perp \neq 0, j_\parallel = 0 \) and either \( p > -1 \) or \( p = j_{\text{thresh}} - 1 \), respectively. (e) and (f) \( j_\perp \neq 0, j_\parallel \neq 0 \) and either \( p > -1 \)
or \( p = \sqrt{J_{\text{thresh}}^2 - j_\|^2} - 1 \), respectively.

**Figure 7:** Non-potential three-dimensional magnetic field structures for situations where the magnitude of the current parallel to the spine is equal to the threshold value \( | p - 1 | \). (a) \( j_\perp \neq 0, j_\| = 0 \) and \( p = 1 \), respectively. (b) and (c) \( j_\perp = 0, j_\| \neq 0 \) and either \( p > -1 \) or \( p = -1 \), respectively. (d) and (e) \( j_\perp \neq 0, j_\| \neq 0 \) and either \( p > -1 \) or \( p = -1 \), respectively.

**Figure 8:** Magnetic field configurations of three-dimensional non-potential neutral points where the magnitude of the current parallel to the spine is greater than the threshold value \( q = 0 \). (a) and (b) \( j_\perp = 0, j_\| \neq 0 \). (c) \( j_\perp \neq 0, j_\| \neq 0 \). (d) \( j_\perp = 0, j_\| \neq 0 \) (xz-plane). (e) \( p = -1, j_\perp = 0, j_\| \neq 0 \). (f) \( p = -1, j_\perp \neq 0, j_\| \neq 0 \).
2D Nulls
\[ B = M \cdot r, \quad r = (x, y)^T, \quad J = (0, 0, j_z) / \mu_0 \]

- **Potential Null**
  - \( |J| = 0 \)
  - M symmetric, two real eigenvalues
  - X-type neutral point, §II.A

- **Non-Potential Nulls**
  - \( |J| > 0 \)
  - M asymmetric, §II.B
    - \( |j_z| < j_{\text{thresh}} \)
    - \( |j_z| = j_{\text{thresh}} \)
    - \( |j_z| > j_{\text{thresh}} \)

- **X-type Neutral Point**
  - real eigenvalues, §II.B.1

- **Anti-parallel Lines**
  - repeated eigenvalues (both zero), §II.B.2

- **O-type Neutral Point**
  - complex conjugate eigenvalues, §II.B.3

Figure 1:
Figure 2:
Figure 3:
Fan plane inclined

Fan plane inclined

Fan plane \perp spine axis

Fan plane \perp spine axis

Potential Nulls
M symmetric
three real eigenvalues
fan plane \perp spine axis,
§IV

Non-Potential Nulls
M asymmetric, §V

Real Eigenvalues
three distinct \(|j| < j_{\text{thresh}}\), §V.A
two equal \(|j| = j_{\text{thresh}}\), §V.B

Complex Eigenvalues
one real, two complex conjugate, §V.C

\(|J| = 0\)

\(|J| > 0\)

\(|j| \leq j_{\text{thresh}}\)

\(|j| > j_{\text{thresh}}\)

\(j_\perp = 0\)

\(|j_\perp| > 0\)

\(j_\perp = 0\)

\(|j_\perp| > 0\)

Figure 4:
\[ \mathbf{B} = (x, py, -(p + 1)z) \ (\mathbf{J} = 0) \]

Figure 5:
\[ B = (x - \frac{1}{2} j_{\parallel} y, \frac{1}{2} j_{\parallel} x + p y, j_{\perp} y - (p + 1) z) \]
\[ |j_{\parallel}| < j_{\text{thresh}} \]

Figure 6:
\[ \mathbf{B} = (x - \frac{1}{2} \hat{j}_y y, \frac{1}{2} \hat{j}_x x + p y, \hat{j}_y y - (p + 1) z) \]
\[ \left| \hat{j}_{} \right| = j_{thresh} \]

(a) \( p = 1, \hat{j}_|| = 0, \hat{j}_\perp = 0.5 \)

(b) \( p = 0.5, \hat{j}_|| = 0.5, \hat{j}_\perp = 0 \)

(c) \( p = \perp 1, \hat{j}_|| = 2, \hat{j}_\perp = 0 \)

(d) \( p = 0.5, \hat{j}_|| = \perp 0.5, \hat{j}_\perp = 1.5 \)

(e) \( p = \perp 1, \hat{j}_|| = 2, \hat{j}_\perp = 3 \)

Figure 7:
\[ \mathbf{B} = (x - \frac{1}{2} j_{\parallel} y, \frac{1}{2} j_{\parallel} x + py, j_{\perp} y - (p + 1) z) \]

\[ |j_{\parallel}| > j_{\text{thresh}} \]

\begin{align*}
(a) & \quad p = 0.5, \; j_{\parallel} = 1, \; j_{\perp} = 0 \\
(b) & \quad p = 1, \; j_{\parallel} = 4, \; j_{\perp} = 0 \\
(c) & \quad p = -0.5, \; j_{\parallel} = 2, \; j_{\perp} = 0.25 \\
(d) & \quad (xz\text{-plane}) \quad p = -0.5, \; j_{\parallel} = 4, \; j_{\perp} = 0 \\
(e) & \quad p = 1, \; j_{\parallel} = 3, \; j_{\perp} = 0 \\
(f) & \quad p = 1, \; j_{\parallel} = 3, \; j_{\perp} = 0.5
\end{align*}

Figure 8:
### Eigenvalues

<table>
<thead>
<tr>
<th>Three real and distinct</th>
<th>The associated eigenvectors ((x_1, x_2, x_3)).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two repeated, one distinct</td>
<td>The three eigenvectors if they exist, or the vectors ((x_1, x'_2, x_3)) which satisfy (Mx_1 = \lambda x_1), (Mx'_2 = \lambda x'_2 + x_1) and (Mx_3 = -2\lambda x_3).</td>
</tr>
<tr>
<td>Two complex conjugate, one real</td>
<td>If the eigenvectors are (x_1, x_2) and (x_3) then the vectors are (x'_1 = (x_1 + x_2) / 2), (x'_2 = -i (x_1 - x_2) / 2) and (x_3).</td>
</tr>
</tbody>
</table>

Table 1:
### Three-Dimensional Null Determining Rules

<table>
<thead>
<tr>
<th>Discriminant</th>
<th>$\lambda I - M$</th>
<th>$M$ symmetric?</th>
<th>$\det(M)$</th>
<th>Type of null</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; 0$</td>
<td>yes $\Rightarrow$ potential</td>
<td>$\neq 0$, $\Rightarrow$ 3D null</td>
<td>Improper radial null</td>
<td></td>
</tr>
<tr>
<td>$(j_{\text{thresh}}^2 - j_{\parallel}^2 &gt; 0)$</td>
<td></td>
<td>$&lt; 0$ positive, $&gt; 0$ negative</td>
<td>(fan - spine)</td>
<td></td>
</tr>
<tr>
<td>no $\Rightarrow$ non-potential</td>
<td></td>
<td>$= 0$, $\Rightarrow$ 2D null</td>
<td>Continuous X-pts</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>($-$ to null line)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$= 0$</td>
<td>yes $\Rightarrow$ potential</td>
<td>$\neq 0$, $\Rightarrow$ 3D null</td>
<td>Proper radial null</td>
<td></td>
</tr>
<tr>
<td>$(j_{\text{thresh}}^2 - j_{\parallel}^2 = 0)$</td>
<td></td>
<td>$&lt; 0$ positive, $&gt; 0$ negative</td>
<td>(fan - spine)</td>
<td></td>
</tr>
<tr>
<td>no $\Rightarrow$ non-potential</td>
<td></td>
<td>$= 0$, $\Rightarrow$ 2D null</td>
<td>Antiparallel lines with null plane</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>($-$ to null line if $j_{\perp} = 0$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$&gt; 0$</td>
<td>no $\Rightarrow$ non-potential</td>
<td>$\neq 0$, $\Rightarrow$ 3D null</td>
<td>Spiral null</td>
<td></td>
</tr>
<tr>
<td>$(j_{\text{thresh}}^2 - j_{\parallel}^2 &lt; 0)$</td>
<td></td>
<td>$&lt; 0$ positive, $&gt; 0$ negative</td>
<td>(fan - spine if $j_{\perp} = 0$)</td>
<td></td>
</tr>
<tr>
<td>no $\Rightarrow$ non-potential</td>
<td></td>
<td>$= 0$, $\Rightarrow$ 2D null</td>
<td>Continuous concentric ellipses</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>($-$ to null line if $j_{\perp} = 0$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>($\neq$ to null line if $j_{\perp} \neq 0$)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>