Alfvén Wave Phase-Mixing and Damping in the Ion Cyclotron Range of Frequencies

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Introduction

- Coronal Heating Problem - $T_{\text{corona}} \sim 10^6 \text{K}$, $T_{\text{Photosphere}} \sim 10^4 \text{K}$.
- Potential heating mechanism - phase-mixing of Alfvén waves
  - Footpoint motions excite MHD waves into corona
  - Density changes across field cause variation in propagation speed $c_A(x)$
  - Phase differences lead to steep amplitude gradients which significantly increase damping rates - originally shown by Heyvaerts and Priest (1983), and more recently by Hood et al. (2002).
- Investigation: How this mechanism is affected by the Hall term?
- Require high frequency waves ($\sim \Omega_i$) - not easily detected
- Possibly may be found in flaring corona, have been observed in situ at Earth's bow shock (Sckopke et al., 1990)
When are the Extra Effects Appropriate?

In an ideal low beta plasma, the Generalised Ohm’s Law becomes:

\[ E + u \times B = \frac{m^-}{ne^2} \frac{\partial j}{\partial t} + \frac{1}{ne} j \times B \]

*What conditions require MHD treatments to include these extra terms?*
When are the Extra Effects Appropriate?

In an ideal low beta plasma, the Generalised Ohm's Law becomes:

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{m^-}{ne^2} \frac{\partial \mathbf{j}}{\partial t} + \frac{1}{ne} \mathbf{j} \times \mathbf{B}$$

What conditions require MHD treatments to include these extra terms?

- Use Faraday and Ampère's Laws to assess relative magnitudes of terms:

  $$\frac{L \omega}{c} : \frac{L \omega}{c} : \frac{\omega}{\omega_{pe}^2} \frac{c}{L} : \frac{|\Omega_e| c}{\omega_{pe}^2 L}$$

- Hall term must be included when $L \sim \delta_i$ (ion skin-depth $\delta_i = c/\omega_{pi} \approx 43\delta_e$), and $\omega \sim \Omega_i$ (ion cyclotron frequency)
Uniform Resistive Hall MHD

- *Ideal* Ohms law in a low beta environment:
  \[ \mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 \]

- Form induction equation and hence evolution equation for transverse perturbation \( B_c = B_z \):
  \[ \frac{\partial^2 B_c}{\partial t^2} = c_A^2 \frac{\partial^2 B_c}{\partial y^2} \]

- Seek wave-like solutions travelling parallel to field \( \sim \exp(i [k y - \omega t]) \), find dispersion relation:
  \[ \omega = \pm c_A k \]
Uniform Resistive Hall MHD

- Ohms law in a low beta environment where \( \omega \sim \Omega_i \) and \( L \sim \delta_i \) (e.g. achieved through *phase-mixing*) becomes:

\[
E + u \times B = \eta j + \frac{1}{ne} j \times B
\]

- Form induction equation and hence evolution equation for transverse perturbations \( B_c = B_z + iB_x \):

\[
\frac{\partial^2 B_c}{\partial t^2} = c_A^2 \frac{\partial^2 B_c}{\partial y^2} + \left( \frac{\eta}{\mu_0} - i c_A \delta_i \right) \frac{\partial^2}{\partial y^2} \left( \frac{\partial B_c}{\partial t} \right)
\]

- Seek wave-like solutions travelling parallel to field \( \sim \exp(i [ky - \omega t]) \), find dispersion relation:

\[
\omega = \pm \frac{c_A \delta_i k^2}{2} \left\{ -1 \pm \sqrt{1 + \frac{4}{k^2 \delta_i^2}} \right\}
\]
Uniform Hall MHD - Distinct Wave Solutions

Limit of $k^2 \delta_i^2 \gg 1$ positive & negative roots inside the bracket yield:

- $\omega \simeq \pm c_A \delta_i k^2$ - whistler wave.
- $\omega \simeq \pm c_A / \delta_i = \pm \Omega_i$ - ion cyclotron (IC) wave.

In the limit where $k^2 \delta_i^2 \ll 1$ we find:

- $\omega \simeq \frac{c_A \delta_i k^2}{2} \pm c_A k$ - a combination of whistler and Alfvén waves.
For \( k^2 \delta_i^2 \ll 1 \), solve for \( B_c(y, t) \) using Fourier transformation for initially Gaussian perturbation:

\[
B_c(y, 0) = B_1 \exp \left[ -\frac{y^2}{2\sigma^2} \right]
\]

\[
B_c(y, t) = \int_{-\infty}^{\infty} f(k) \exp \left[ i(ky - \omega t) \right] dk
\]
For $k^2 \delta_i^2 \ll 1$, solve for $B_c(y, t)$ using Fourier transformation for initially Gaussian perturbation:

$$B_c(y, t) = \frac{B_1}{2\sqrt{1 + \left(\frac{\eta}{\mu_0} - i c_A \delta_i\right) t / \sigma^2}} \exp\left(-\frac{(y \pm c_A t)^2}{2 \left[ \sigma^2 + \left(\frac{\eta}{\mu_0} - i c_A \delta_i\right) t \right]}ight)$$

Sought to recover analytical results using Lare2d (Arber et al., 2001):

- Equilibrium $B$ field along $y$
- Gaussian perturbation along $y$ in $B_z$
- Transverse density profile applied in $x$
Without the Hall term, we recover expected shear Alfvén wave solution:
Uniform Hall MHD simulations - $k^2 \delta^2_i \ll 1$ limit

Shear wave solution modified by presence of Hall term:
Wave Dissipation in Uniform Plasma

- Using analytical expression $B_c(y, t)$, derive expression for total perturbed magnetic field energy evolution in Hall MHD ($k^2 \delta_i^2 \ll 1$) which is identical to that found in the MHD limit:

$$\mathcal{E}_{B_c}^{\text{Hall}} = \frac{1}{2\mu_0} \int B_c B_c^* dy$$

$$= \frac{\sqrt{\pi} \sigma B_1^2}{4\mu_0 (1 + \eta t/\mu_0 \sigma^2)^{1/2}} \left\{ 1 + \exp \left( -\frac{c_A^2 t^2}{\sigma^2 + \eta t/\mu_0} \right) \right\}$$

- Obtain two expressions for $k^2 \delta_i^2 \gg 1$ differing from the MHD result:

$$\mathcal{E}_{B_c}^{\text{w}} = \frac{\sqrt{\pi} \sigma B_1^2}{2\mu_0} \left( 1 + \frac{2\eta t}{\mu_0 \sigma^2} \right)^{-\frac{1}{2}}$$

$$\mathcal{E}_{B_c}^{\text{ic}} = \frac{\sqrt{\pi} \sigma B_1^2}{2\mu_0} \exp \left( -\frac{2\eta t}{\mu_0 \delta_i^2} \right)$$
Energy Equipartition in Uniform Plasma

- Simulations confirm *no difference* between $k^2 \delta_i^2 \ll 1$ and MHD results, in agreement with analysis.
- Cannot simulate $k^2 \delta_i^2 \gg 1$ limit, gradual increase of $\delta_i$ begins to move pulse energy towards dominant whistler energy contribution (★)
Phase-Mixing Studies

- Heyvaerts and Priest (1983) asserted that standing waves, undergoing phase-mixing in an MHD plasma damp at a rate \( \propto \exp\left(-\epsilon t^3\right) \).
- Hood et al. (2002), consider only a single pulse, and obtain a decay rate \( \propto \left(\sigma^2 + 1/3(c_{A'})^2 \eta/\mu_0 t^3\right)^{-1/2} \).
- We numerically recover amplitude damping rate of Hood et al. (2002) (provided weak damping and strong phase-mixing criterion are met) in both MHD & Hall MHD regimes:
Phase-mixing *dominates behaviour* at locations of steepest density gradients in both MHD and Hall MHD.

Elsewhere, clear differences in behaviour between MHD and Hall MHD cases:
Phase-mixing increases conversion rate of magnetic → internal energy

Can compare the energy balance of uniform (analytical and numerical) and non-uniform Hall MHD plasmas using Lare2d (here for $k^2 \delta_i^2 \ll 1$ limit):

![Graph showing energy balance over time]

- Magnetic (analytical)
- Magnetic (numerical)
- Kinetic (numerical)
- Internal (numerical)
Conclusions

- Investigated the effect of the Hall term on the propagation of initially Gaussian pulse.
- Describe analytically the evolution of the pulse and its associated magnetic energy in a uniform equilibrium plasma.
- Find that the energy dissipation rates are *identical* for MHD and Hall MHD when $k^2 \delta_i^2 \ll 1$, but are different in the $k^2 \delta_i^2 \gg 1$ limit.
- Recover pulse amplitude damping rate given by (Hood et al., 2002) for phase-mixing cases, in both Hall MHD and MHD (for the $k^2 \delta_i^2 \ll 1$ limit) at locations of steepest density gradient.
- Seek to investigate how increasing the parameter $k^2 \delta_i^2$ affects the equipartition of energy in the simulations.

