Alfvén Wave Phase-Mixing and Damping in the Ion Cyclotron Range of Frequencies

J. Threlfall\textsuperscript{1} (I. De Moortel\textsuperscript{1} & K. G. McClements\textsuperscript{2})

(1) University of St Andrews

(2) EURATOM/CCFE (Culham Centre for Fusion Energy)

jamest@mcs.st-and.ac.uk

May 4th, 2011
A long time ago in a contents page far, far away...

The Hall Term - A Phantom Menace?
  Introduction, conditions on Hall MHD..

Attack of the Uniform Hall MHD
  Analytical solutions
  Uniform Hall MHD simulations (initially Gaussian pulse)
  Recovery of energy dissipation rates

Revenge of the Non-Uniform Hall MHD
  Effect on phase-mixing damping rate

Boundary-Driven Phase-Mixing - A New Hope?
  Simulations of phase-mixed sinusoidal boundary motions

The X-Point Strikes Back
  Current/Future work: pulse & boundary driven, reconnection implications?

Return of the Conclusions
Introduction

- Coronal heating problem - \( T_{\text{corona}} \sim 10^6 \text{ K}, \ T_{\text{photosphere}} \sim 10^4 \text{ K}. \)

- Potential heating mechanism - phase-mixing of Alfvén waves (Heyvaerts and Priest, 1983)
  - Footpoint motions excite MHD waves into corona
  - Density changes across field cause variation in propagation speed \( c_A(x) \)
  - Phase differences lead to steep amplitude gradients which significantly increase damping rates

- **Investigation:** How is this MHD mechanism affected by the Hall term?
Extending MHD - when is this appropriate?

In an ideal low beta plasma, the Generalised Ohm’s law becomes:

\[ E + u \times B = \frac{m^-}{ne^2} \frac{\partial j}{\partial t} + \frac{1}{ne} j \times B \]

What conditions require MHD treatments to include these extra terms?
Extending MHD - when is this appropriate?

In an ideal low beta plasma, the Generalised Ohm’s law becomes:

$$ E + u \times B = \frac{m^-}{ne^2} \frac{\partial j}{\partial t} + \frac{1}{ne} j \times B $$

**What conditions require MHD treatments to include these extra terms?**

- Use Faraday and Ampère’s laws to assess relative magnitudes of terms:

  $$ \frac{L\omega}{c} : \frac{L\omega}{c} : \frac{\omega}{\omega_{pe}^2} \frac{c}{L} : \frac{|\Omega_e|}{\omega_{pe}^2} \frac{c}{L} $$

- **Hall term must be included** when $L \sim \delta_i$ (ion skin-depth $\delta_i = c/\omega_{pi} \approx 43\delta_e$), and $\omega \sim \Omega_i$ (ion cyclotron frequency)
Hall MHD - Is It Important???

- Hall term relevant for small lengthscales ($\sim \delta_i$) and high frequencies ($\sim \Omega_i$)
- Difficult to detect waves with $\omega \sim \Omega_i$ - thought to occur in flaring corona, have been observed at Earth’s bow shock (Sckopke et al., 1990).
- Very important for reconnection: GEM Challenge (Birn et al., 2001) tested various simulation models, found “reconnection rate .. is essentially the same for all models which include the Hall term” (Birn and Priest, 2007).
Uniform MHD/Hall MHD dispersion relations

- Ideal Ohms law in a low beta environment:

\[ \mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 \]

- Form induction equation and hence evolution equation for transverse perturbation \( B_z \):

\[ \frac{\partial^2 B_z}{\partial t^2} - c_A^2 \frac{\partial^2 B_z}{\partial y^2} = 0 \]

- Seek wave-like solutions travelling parallel to field \( \sim \exp (i [ky - \omega t]) \), recover ideal dispersion relation:

\[ \omega = \pm c_A k \]
Uniform MHD/Hall MHD dispersion relations

- Ohms law in a low beta environment where $\omega \sim \Omega_i$ and $L \sim \delta_i$ becomes:

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{j} + \frac{1}{ne} \mathbf{j} \times \mathbf{B}$$

- Form induction equation and hence evolution equation for transverse perturbations $B_c = B_z + iB_x$:

$$\frac{\partial^2 B_c}{\partial t^2} - c_A^2 \frac{\partial^2 B_c}{\partial y^2} = \left( \frac{\eta}{\mu_0} - i c_A \delta_i \right) \frac{\partial^2}{\partial y^2} \left( \frac{\partial B_c}{\partial t} \right)$$

- Seek wave-like solutions travelling parallel to field $\sim \exp (i [ky - \omega t])$, recover ideal dispersion relation:

$$\omega = \pm \frac{c_A \delta_i k^2}{2} \left\{ -1 \pm \sqrt{1 + \frac{4}{k^2 \delta_i^2}} \right\}$$
Uniform Hall MHD - distinct wave solutions

- Limit of \( k^2 \delta_i^2 \gg 1 \) positive & negative roots inside the bracket yield:
  - \( \omega \simeq \pm c_A \delta_i k^2 \) - whistler wave.
  - \( \omega \simeq \pm c_A / \delta_i = \pm \Omega_i \) - ion cyclotron (IC) wave.

- In the limit where \( k^2 \delta_i^2 \ll 1 \) we find:
  - \( \omega \simeq \frac{c_A \delta_i}{2} \pm c_A k \) - a combination of whistler and Alfvén waves.
• For $k^2 \delta_i^2 \ll 1$, solve for $B_c(y, t)$ using Fourier transformation for initially Gaussian perturbation:

\[ B_c(y, 0) = B_1 \exp \left[ -\frac{y^2}{2\sigma^2} \right] \]

\[ B_c(y, t) = \int_{-\infty}^{\infty} f(k) \exp \left[ i(ky - \omega t) \right] dk \]
Uniform Hall MHD - distinct wave solutions (contd.)

- For $k^2 \delta_i^2 \ll 1$, solve for $B_c(y, t)$ using Fourier transformation for initially Gaussian perturbation:

$$B_c(y, t) = \frac{B_1}{2 \sqrt{1 + \left( \frac{n}{\mu_0} - ic_A \delta_i \right) t/\sigma^2}} \exp \left( -\frac{(y \pm c_A t)^2}{2 \left[ \sigma^2 + \left( \frac{n}{\mu_0} - ic_A \delta_i \right) t \right]} \right)$$

- Simulated using Lare2d (Arber et al., 2001):

  - Equilibrium $B$ field along $y$
  - Gaussian perturbation along $y$ in $B_z$
  - Optional transverse density profile (in $x$)
Benchmark - Recovery of Uniform MHD Solution

Without the Hall term, we recover expected shear Alfvén wave solution:
Uniform Hall MHD simulations - $k^2 \delta_i^2 \ll 1$ limit

Shear wave solution modified by presence of Hall term:
Wave dissipation in uniform plasma

- Using analytical expression $B_c(y, t)$, derive expression for total perturbed magnetic field energy evolution in Hall MHD ($k^2 \delta_i^2 \ll 1$) which is identical to that found in the MHD limit:

$$\mathcal{E}_{B_c}^{\text{Hall}} = \frac{1}{2\mu_0} \int B_c B_c^* dy = \frac{\sqrt{\pi} \sigma B_1^2}{4\mu_0 (1 + \eta t / \mu_0 \sigma^2)^{1/2}} \left\{ 1 + \exp \left( -\frac{c_A^2 t^2}{\sigma^2 + \frac{\eta}{\mu_0} t} \right) \right\}$$

- Obtain two expressions for $k^2 \delta_i^2 \gg 1$ differing from the MHD result:

$$\mathcal{E}_{B_c}^w = \frac{\sqrt{\pi} \sigma B_1^2}{2\mu_0} \left( 1 + \frac{2\eta t}{\mu_0 \sigma^2} \right)^{-1/2}$$

$$\mathcal{E}_{B_c}^{ic} = \frac{\sqrt{\pi} \sigma B_1^2}{2\mu_0} \exp \left( -\frac{2\eta t}{\mu_0 \delta_i^2} \right)$$
Non-uniform plasma studies

- Numerically recover identical MHD/long wavelength Hall MHD energy dissipation rates with uniform density.

- In order to investigate phase-mixing, introduce density gradient \(0 \geq x \geq 1\) with the form:

\[
\rho(x) = \frac{1}{\sqrt{1 + \alpha - \alpha \cos(2\pi x)}}
\]

(N.B. \(\rho(0) = \rho(1) = 1\), and the central density enhancement is controlled by parameter \(\alpha\)).
Non-uniform plasma studies - $k^2 \delta_i^2 \ll 1$

- $k^2 \delta_i^2 \ll 1$ cases exhibit weaker damping than MHD cases, for densities which are neither uniform nor sharply varying.
Non-uniform plasma studies - $k^2 \delta_i^2 \sim 1$

- Effect is magnified in higher skin depth $k^2 \delta_i^2 \sim 1$ Hall MHD cases.
Causes of reduction in non-uniform damping rate?

- Dispersive leading edge whistler component increases speed with skin depth.
Causes of reduction in non-uniform damping rate?

- Slices through pulse display weakened amplitude gradients across the field in Hall MHD cases:

- In Hall MHD, pulse energy is dispersed along the field, reducing transverse gradients, reducing visco-resistive damping.
Boundary Driven Simulations - Amplitude Decay Rates

- Replace initial value problem with boundary driver, $B_z, V_z \propto A \sin(\omega t)$.

- Initially recovered decay rate originally found by Heyvaerts and Priest (1983) in MHD using Lare2d.
Boundary Driven Simulations - Amplitude Decay Rates

- As with initially Gaussian pulse results, Hall MHD causes energy dispersion.
- The presence of a range of wavenumbers & speeds causes multiple locations where phase differences generate amplitude damping, which (in total) is weaker than MHD, due to smaller wave amplitudes throughout the domain.
Boundary Driven Simulations - Energy Damping Rates

- Focusing on the energy decay rates, once again phase-mixing appears to be attenuated by the presence of the Hall term.
Boundary Driven Simulations - Fractional Energies

- Fractional energies show clear similarities to Gaussian pulse energy cases.
X-point Geometry

- Previous phase-mixing studies focused on variation in Alfvén speed caused by density gradient.
- In studying pulse/wave behaviour near an X-point, we obtain phase-mixing caused by local field variations at constant density.
- Equilibrium field given by $\mathbf{B} = B_0[-x, y, 0]$
Pulses near an X-point

- 2D Gaussian pulse studies
- Initial pulse shape $B_z \propto A \exp \left( -\frac{(x - 2.5)^2}{\sigma_x} \right) \exp \left( -\frac{(y - 2.5)^2}{\sigma_y} \right)$
- Find that increasing skin depth $\delta_i$ allows significant amount of pulse energy to cross separatrices
Pulses near an X-point (energies)

- Pulse energies are identical for $k^2 \delta_i^2 \ll 1$ and MHD cases as expected.
- Increasing skin depth weakens magnetic-internal energy conversion, **but** effect appears to diminish for highest skin depth case simulated - effects of localised phase-mixing near the null?
Wave/X-Point Interaction

- Have recently begun to investigate shear waves in x-point geometry.
- Boundary driver, $v_z, B_z \propto A \sin(\omega t) \exp(-y^2/\delta y)$ in order to focus on wave interacting with x-point.
Wave/X-Point Interactions (contd..)

- Preliminary evidence suggests that Hall regime causes very minor differences in flux, implying extremely small levels of reconnection.
- No visible change in field line connectivity - influence of periodic boundaries?
- Wide scope for further investigation.....
That's no moon, it's some **Key Points!**

- Hall term important for $L \sim \delta_i$ and $\omega \sim \Omega_i$.
- Crucial differences with Hall MHD - *whistler and ion cyclotron waves*, which are *circularly polarised*.
- Energy of initially Gaussian pulse *identical* in MHD and $k^2 \delta_i^2 \ll 1$ for a *uniform or phase-mixing dominated* plasma.
- Dispersive whistler effects reduce effectiveness of phase-mixing, by reducing amplitude gradients across the field.

**Further discussion found in Threlfall et al. (2011, A&A 525 A155)**

- Appears to be a generic result, as also seen in phase-mixing of boundary driven waves and for pulses near an x-point.
- Currently studying wave/x-point interactions with a view to studying Hall MHD and reconnection?


