4th SOLAIRE School

Solar MHD, Reconnection Theory, Flares and CMEs

Equilibria and magnetic field extrapolations

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Coronal equilibrium

Three characteristic times are important: τ_{eq} time of equilibrium, τ_A Alfvén transit time, τ_{rec} reconnection time. Typically, the Alfvén transit time is several minutes and the reconnection time is few seconds.

$$\tau_{eq} \gg \tau_A \gg \tau_{rec}$$

The equilibrium can relax to a minimum energy state (*Woltjer 1958*)

Time evolution can be describe by a series of linear force-free equilibria (*Heyvaerts and Priest 1984*)

$$\tau_{eq} > \tau_A \gg \tau_{rec}$$

The equilibrium will more likely relax to a nonlinear force-free field

We have shown that we can successfully describe the time evolution of an active region with a time series of nonlinear force-free equilibria with a time interval of 15 min (*Antiochos 1987; Régnier and Canfield 2006*)

Equilibrium equations

Assuming an equilibrium, three forces are acting on the plasma: magnetic forces, plasma pressure gradients, and gravitational force

Magneto-hydrostatic equilibrium

$$-\vec{\nabla} p + \rho \vec{g} + \vec{j} \wedge \vec{B} = \vec{0}$$

Hydrostatic equilibrium

$$-\vec{\nabla} \mathrm{p} + \rho \vec{\mathrm{g}} = \vec{0}$$

Force-free equilibrium

$$\vec{j} \wedge \vec{B} = \vec{0}$$

Potential Field	Linear Force-free Field	Nonlinear Force-free Field
$\vec{j} = \vec{0}$	$\vec{j} = \frac{\vec{\nabla} \wedge \vec{B}}{\mu_0} = \alpha \vec{B}$	$\vec{j} = \alpha(\vec{r}) \vec{B}$

Hydrostatic equilibrium

1. ***Equation***
2. ***Examples***

Equation

$$-\vec{\nabla}p + \rho\vec{g} = \vec{0}$$

The solution of this equation is given by

$$p = p_0 \exp\left(-\int_0^z \frac{1}{\Lambda(z)} dz \right)$$

where $\Lambda(z) = \frac{k_B T(z)}{m g(z)}$ is the pressure scale height

For an isothermal atmosphere, the pressure scale height is a constant:

In the photosphere: ~200 km

In the corona: ~50 Mm

Force-free magnetic fields

1. ***Equations and boundary conditions***
2. ***General properties***

Force-free fields

We consider that the corona is dominated by the magnetic field (low-β plasma) and therefore the coronal magnetic field is assumed to be force-free.

Force-free equilibrium $\vec{j} \wedge \vec{B} = \vec{0}$

$$\vec{j} = \vec{0}$$

Potential Field

$$\vec{j} = \frac{\vec{\nabla} \wedge \vec{B}}{\mu_0} = \alpha \vec{B}$$

Linear Force-free Field

$$\vec{j} = \alpha(\vec{r})\vec{B}$$

Nonlinear Force-free Field

Properties of force-free magnetic fields

Necessary integral properties of a force-free field (not sufficient) from *Molodensky* (1969) and *Aly* (1989) deduced from the Maxwell stress tensor.

No magnetic force

$$\int_{\partial\Omega} B_x B_z \, \mathrm{d}\sigma = \int_{\partial\Omega} B_y B_z \, \mathrm{d}\sigma = 0 ,$$

$$\int_{\partial\Omega} (B_x^2 + B_y^2) \, \mathrm{d}\sigma = \int_{\partial\Omega} B_z^2 \, \mathrm{d}\sigma ,$$

No magnetic torque

$$\int_{\partial\Omega} \begin{Bmatrix} x \\ y \end{Bmatrix} (B_x^2 + B_y^2) \, \mathrm{d}\sigma = \int_{\partial\Omega} \begin{Bmatrix} x \\ y \end{Bmatrix} B_z^2 \, \mathrm{d}\sigma ,$$

$$\int_{\partial\Omega} y B_x B_z \, \mathrm{d}\sigma = \int_{\partial\Omega} x B_y B_z \, \mathrm{d}\sigma ,$$

Molodensky (1969), Aly (1989)

Magnetic helicity

Relative magnetic helicity (Finn & Antonsen 1985, Berger & Field 1984):

$$\Delta H_m = \int_\Omega (\vec{A} + \vec{A}_0) \cdot (\vec{B} - \vec{B}_0) \; d\Omega$$

where **A** is the vector potential associated to **B**, and the index 0 corresponds to a reference field (usually taken to be the potential field).

If $\vec{B} = \vec{B}_{cl} + \vec{B}_{ref}$ then

$$H_{self} = H_m(\vec{B}_{cl}, \vec{A}_{cl}) = \int_\Omega \vec{A}_{cl} \cdot \vec{B}_{cl} \; d\Omega$$

$$H_{mut} = 2H_m(\vec{B}_{cl}, \vec{A}_{ref}) = 2 \int_\Omega \vec{A}_{ref} \cdot \vec{B}_{cl} \; d\Omega$$

H_{self} represents the twist and shear of confined flux bundles

H_{mut} represents the intertwining of field lines as well as the large scale twist

Magnetic Energy

Several quantities are of interest to understand the storage and release of energy:

- the total magnetic energy of a magnetic configuration: $E_m = \int_V \frac{B^2}{8\pi}\, dV$
- the free magnetic energy budget: $\Delta E_m = E_m^{nlff} - E_m^{pot}$
- the magnetic energy density along the vertical axis
- the time evolution of those quantities

Ordering:

$$E_{pot} < E_{nlff} < E_{open}$$

$$E_{pot} < E_{lff}\left(\left|\Delta H_m\right| < \left|\Delta H_m^{nlff}\right|\right) < E_{nlff} < E_{lff}\left(\left|\Delta H_m\right| > \left|\Delta H_m^{nlff}\right|\right) < E_{open}$$

E_{open}: an upper bound for the magnetic energy

Aly-Sturrock conjecture as stated by Aly (1984, ApJ, 283, 349):

> "In an infinite arbitrary domain D and for a given distribution of B_n on ∂D, W_{open} is the least upper bound for the energy W of the regular FFF decaying sufficiently fast to zero at infinity."

The open magnetic field is obtained by considering only one polarity at the bottom boundary layer (changing the sign of the opposite polarity) and by computing the resulting potential field.

The open field magnetic energy is about 1.7-2.2 times the magnetic energy of the potential field (Amari et al. 2000).

Virial theorem

For a force-free, the magnetic energy in a volume V surrounded by a surface S can derived from the *virial theorem*:

$$E_m = \frac{1}{\mu_0} \int_S [(\vec{r}.\vec{B})\vec{B} - \frac{1}{2} B^2 \vec{r}].d\vec{S}$$

Note that the normal to the surface S is pointing outwards

Potential and linear force-free magnetic fields

1. ***Equations and boundary conditions***
2. ***Fourier transform: Gary's method***
3. ***PFSS***

    3.1 Equations

    3.2 Applications

Equation

Potential field $$\Delta\vec{B} = \vec{0}$$

Linear force-free field $$\left(\Delta + \alpha^2\right)\vec{B} = \vec{0}$$

Boundary conditions

- Potential field: normal component of the magnetic field

- Linear force-free field: normal component of the magnetic field and **<u>a guess</u>** for α

Fourier transform: Gary's method (1989)

Fourier transform of magnetic components

$$B_i(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b_i^0(u,v)\, e^{-kz+2\pi iux+2\pi ivy}\,du\,dv$$

Boundary condition on the photosphere

$$b_z^0(u,v) = \int\int B_z(x',y',z=0)\, e^{-2\pi iux'-2\pi ivy'}\,dx'\,dy'$$

Fourier equations on the photosphere (same at z)

$$2\pi iub_y^0 - 2\pi ivb_x^0 = \alpha b_z^0(u,v,0)$$

$$-kb_x^0 - 2\pi iub_z^0 = \alpha b_y^0(u,v,0)$$

$$2\pi ivb_z^0 + kb_y^0 = \alpha b_x^0$$

Condition to get non trivial solutions $k = \pm\left[4\pi^2(u^2+v^2) - \alpha^2\right]^{1/2}$

Important limit $\alpha \le \dfrac{2\pi}{L}$ where L is the characteristic length of the box

PFSS: Potential Field Source Surface

Potential in spherical harmonics

$$\Phi(r,\theta',\phi) = \sum_{\ell,m}\left[A_\ell^m r^\ell + B_\ell^m r^{-(\ell+1)}\right] Y_\ell^m(\theta',\phi)$$

where

$$\begin{cases} Y_\ell^m(\theta',\phi) = C_\ell^m P_\ell^m(\cos\,\theta')\, e^{im\phi} \\ C_\ell^m = (-1)^m \left[\dfrac{2\ell+1}{4\pi}\dfrac{(\ell-m)!}{(\ell+m)!}\right]^{\frac{1}{2}} \end{cases}$$

Corresponding magnetic field components

$$B_r = -\sum_{\ell,m} Y_\ell^m [A_\ell^m \ell r^{\ell-1} - B_\ell^m(\ell+1) r^{-(\ell+2)}]$$

$$B'_\theta = -\frac{1}{r\ \sin\,\theta'}\sum_{\ell,m} Y_\ell^m \{R_\ell^m(\ell-1)[A_{\ell-1}^m r^{\ell-1} + B_{\ell-1}^m r^{-\ell}] - R_{\ell+1}^m(\ell+2)[A_{\ell+1}^m r^{\ell+1} + B_{\ell+1}^m r^{-(\ell+2)}]\},$$

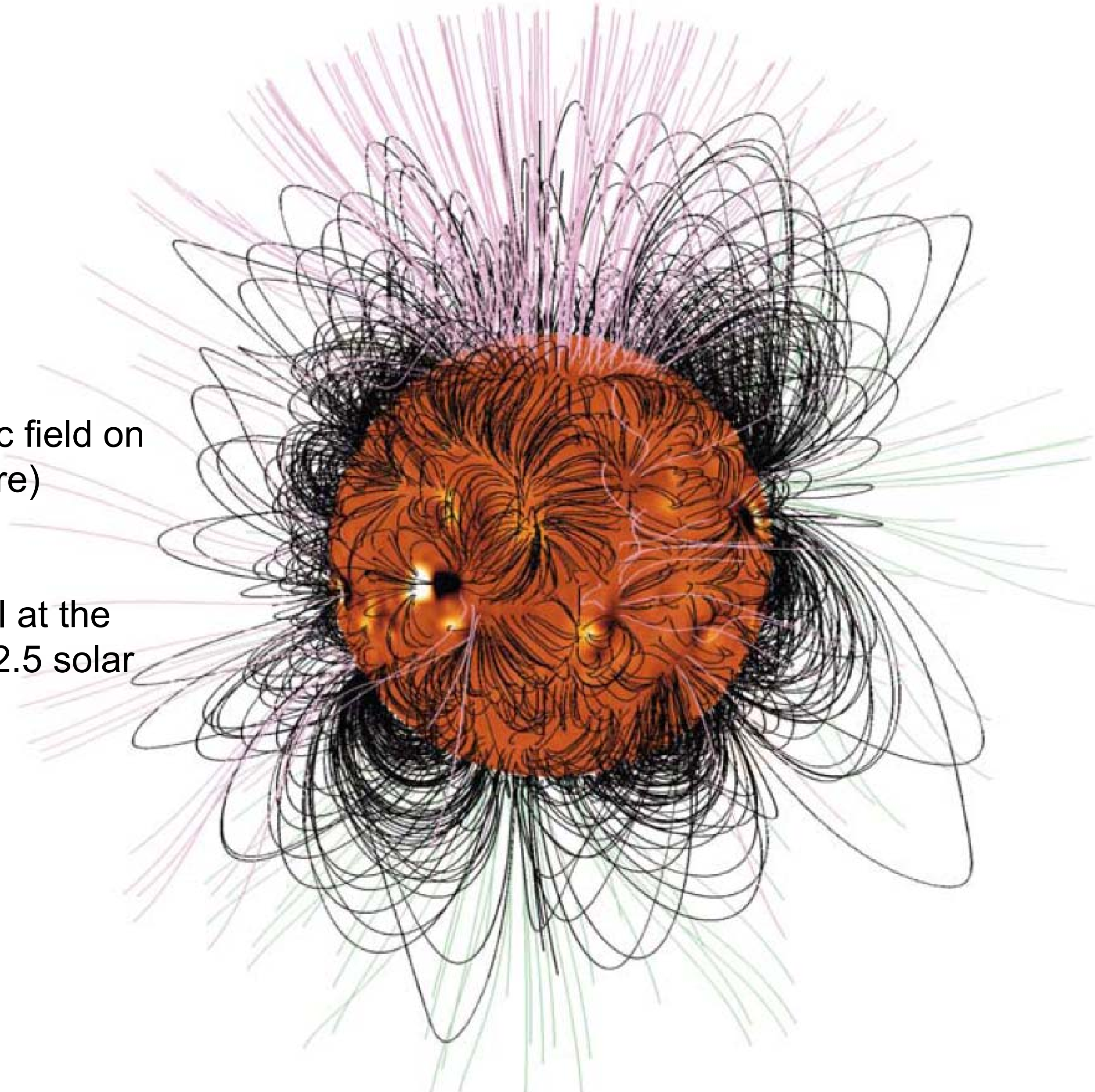
$$B_\phi = -\frac{1}{r\ \sin\,\theta'}\sum_{\ell,m} imY_\ell^m [A_\ell^m r^\ell + B_\ell^m r^{-(\ell+1)}]$$

PFSS: Potential Field Source Surface

Boudary conditions

Synoptic map of the radial component of the magnetic field on the photosphere (full sphere)

The magnetic field is radial at the source surface located at 2.5 solar radii



Non linear force-free magnetic fields

1. ***Equation and boundary conditions***
2. ***Analytical solutions***
3. ***NLFFF extrapolations***

   3.1 Methods

   3.2 Comparison and metrics

Analytical solutions

Some analytical and semi-analytical solutions:

- analytical solutions in cylindrical, spherical and toroidal coordinates have been derived (e.g. Chandrasekhar 1956, Gold-Hoyle 1960, Buck 1965, Low 1973, Titov and Démoulin 1999, Török et al. 2004);

- applications to thin twisted flux tubes in the corona and to magnetic clouds have been performed with the Gold & Hoyle solutions;

$$B_z(r) = \frac{B_0}{1+q^2r^2}, \qquad B_\theta(r) = \frac{B_0 q r}{1+q^2r^2}, \qquad \alpha(r) = \frac{2q}{1+q^2r^2}$$

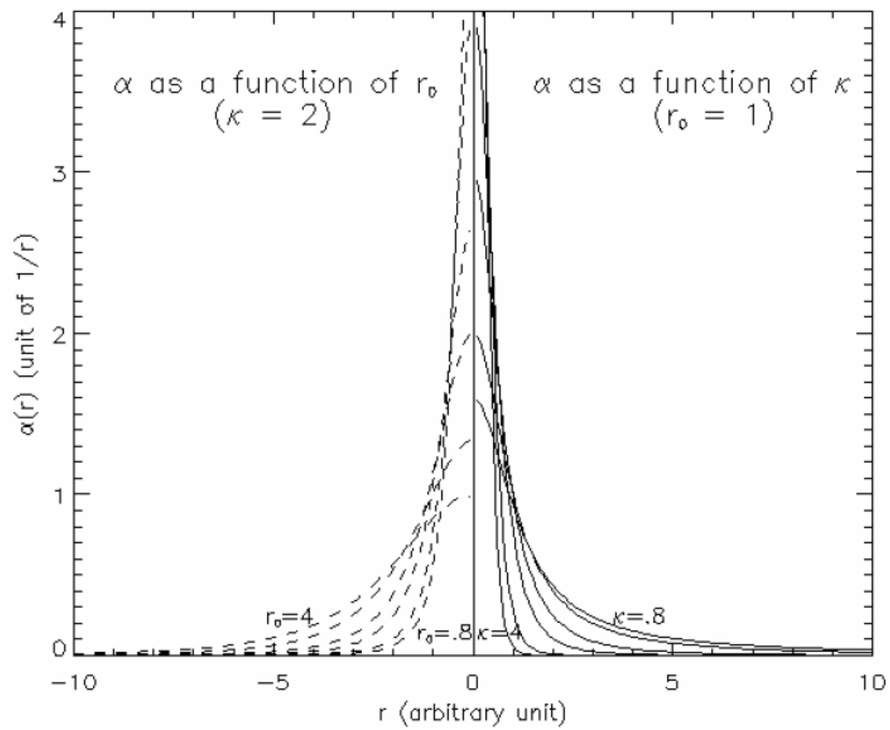
- well-known semi-analytical solutions were derived by Low & Lou (1990); these solutions have been used to test nonlinear force-free reconstruction techniques in Cartesian coordinates.

Analytical solutions

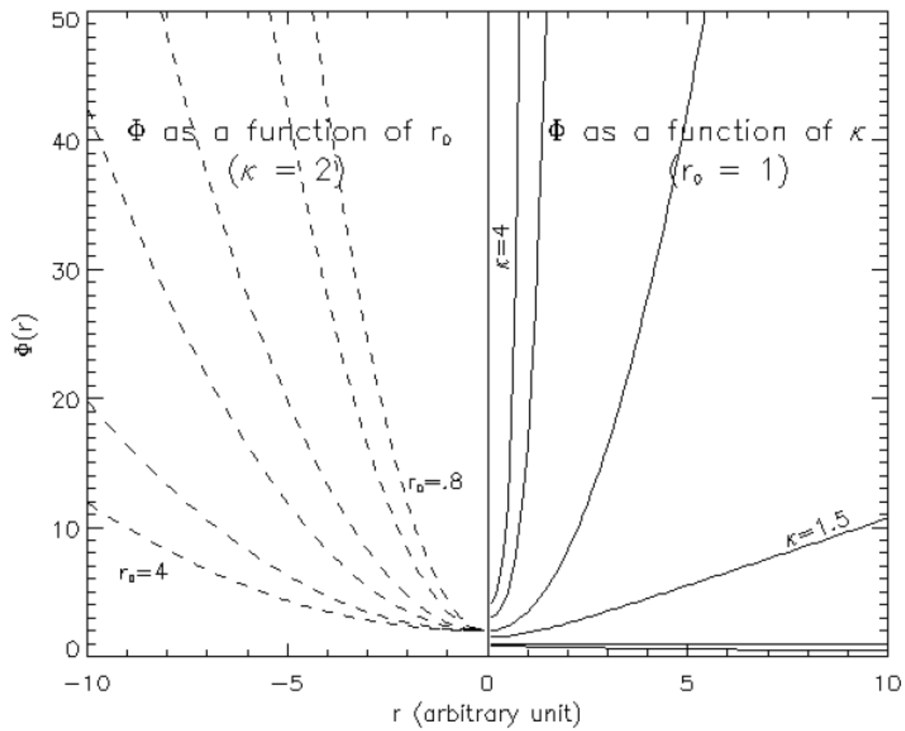
$$B_z(r) = \frac{B_0}{(1+\kappa\frac{r^2}{r_0^2})^\kappa}$$

$$B_\theta(r) = \pm\frac{B_0}{\sqrt{\kappa(2\kappa-1)}}\frac{r_0}{r}\left[1-\frac{1+2\kappa^2\frac{r^2}{r_0^2}}{\left(1+\kappa\frac{r^2}{r_0^2}\right)^{2\kappa}}\right]^{\frac{1}{2}}$$

$$\alpha(r) = \pm\frac{2\kappa^2\sqrt{\kappa(2\kappa-1)}\,r^2}{r_0^3\left(1+\kappa\frac{r^2}{r_0^2}\right)\left[\left(1+\kappa\frac{r^2}{r_0^2}\right)^{2\kappa}-\left(1+2\kappa^2\frac{r^2}{r_0^2}\right)\right]^{\frac{1}{2}}}$$



Twist



NLFFF methods

Vertical integration method (Wu et al. 1990, Démoulin et al. 1992, Song et al. 2006)

Grad & Rubin method (Grad and Rubin 1958, Sakurai 1981, Aly 1988, Amari et al. 1997, 1999, Wheatland 2004, Amari et al. 2006, Inhester and Wiegelmann 2006, …)

Optimization method (Pridmore-Brown et al. 1981, Wheatland et al. 2000, Wiegelmann et al. 2003)

Evolutionary techniques (Mikic and McClymont 1994)

"Stress-and-Relax" technique (Roumeliotis 1996)

Boundary element method (Yan and Sakurai 2000, Li et al. 2004)

Magneto-frictional method (Yang et al. 1986, van Ballegooijen et al. 2000, Mackay et al. 2000, 2001, Valori et al. 2005)

Finite element method (Amari et al. 2006)

(…)

NLFFF methods

Method	boundary conditions at the bottom	boundary conditions on the other sides	initial state	preprocess and comments
Grad & Rubin by Amari et al. (99) by Sakurai and Wheatland by Inhester	 B_z and $\alpha^{\pm}$ " " "	 closed " no B_n unchanged	 potential " " "	 smooth α no no no
Optimization by Wheatland et al. by Wiegelmann by Mc Tiernan	 B_x, B_y, B_z " "	 B_x, B_y, B_z " periodic	 potential " "	 no minimizing forces and torques no
Vertical integration by Wu et al. by Démoulin by Song et al.	 B_x, B_y, B_z " "	 no " "	 no " "	 no smoothing function smooth functions, asymptotic expansion
Evolutionary technique by Mikic et al.	B_z, J_z	closed	potential	external circuit
Stress and relax by Roumeliotis	B_x, B_y, B_z	no	potential	stress (matching B_t on bndary) relax (new B with correct B_t)
Boundary elements by Yan and Sakurai	B_x, B_y, B_z	closed	no	no
Finite elements by Amari et al.	B_z, $\alpha^{\pm}$	B_z, $\alpha^{\pm}$	potential	smooth α

Optimization vs Grad-Rubin

Problem To find the magnetic configuration the closest to a nonlinear force-free and divergence free magnetic field which matches all components of the magnetic field at the boundaries. This is an *ill-posed boundary value problem*

Minimization of a functional L:

$$L = \int_V \left[B^{-2} \left|(\nabla \times \mathbf{B}) \times \mathbf{B}\right|^2 + \left|\nabla \cdot \mathbf{B}\right|^2 \right] d^3V$$

Following Wheatland et al. (2000):

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -2 \int_V \kappa F^2 \mathrm{d}V$$

where
$$\mathbf{F} = \nabla \times (\boldsymbol{\Omega} \times \mathbf{B}) - \boldsymbol{\Omega} \times (\nabla \times \mathbf{B}) - \nabla(\boldsymbol{\Omega} \cdot \mathbf{B}) + \boldsymbol{\Omega}(\nabla \cdot \mathbf{B}) + (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega})\mathbf{B},$$

and $\boldsymbol{\Omega} = B^{-2}\left[(\nabla \times \mathbf{B}) \times \mathbf{B} - (\nabla \cdot \mathbf{B})\mathbf{B}\right].$

The iterative process starts from a potential field, then solves the above equation, and updates the new magnetic field configuration following:

$$\mathbf{B}^{(k+1)} = \mathbf{B}^{(k)} + \mathbf{F}^{(k)} \Delta t$$

and increases t until the functional L is minimized.

Optimization vs Grad-Rubin

Problem To find the nonlinear force-free field associated with those boundary conditions corresponding to a *well-posed boundary value problem*

The Grad-Rubin (1958) scheme separates the nonlinear equations into two linear systems of equations (Aly 1988, Amari et al. 1999):

Transport of α along field lines (hyperbolic):

$$\boldsymbol{B}^{(n)} \cdot \nabla \alpha^{(n)} = 0 \quad \text{in} \quad \Omega,$$

$$\alpha^{(n)}|_{\delta\Omega^{\pm}} = h,$$

Updating the 3d magnetic field (elliptic):

$$\nabla \wedge \boldsymbol{B}^{(n+1)} = \alpha^{(n)} \boldsymbol{B}^{(n)} \quad \text{in} \quad \Omega,$$

$$\nabla \cdot \boldsymbol{B}^{(n+1)} = 0 \quad \text{in} \quad \Omega,$$

$$B_z^{(n+1)}|_{\delta\Omega} = g,$$

$$\lim_{|r|\to\infty} |\boldsymbol{B}| = 0.$$

Properties

Existence and uniqueness of solution for small α (Bineau 1972) in simply connected domains; extended to multiple connected domains (Boulmezaoud and Amari 2000);

Sakurai (1981) and Wheatland (2004) schemes use the magnetic field (*on the left*)

Amari et al. (1997, 1999), Inhester and Wiegelmann (2006) and Amari et al. (2006) use a scheme based on the vector potential to preserve div.B;

Self-consistent solutions (Wheatland and Régnier 2009)

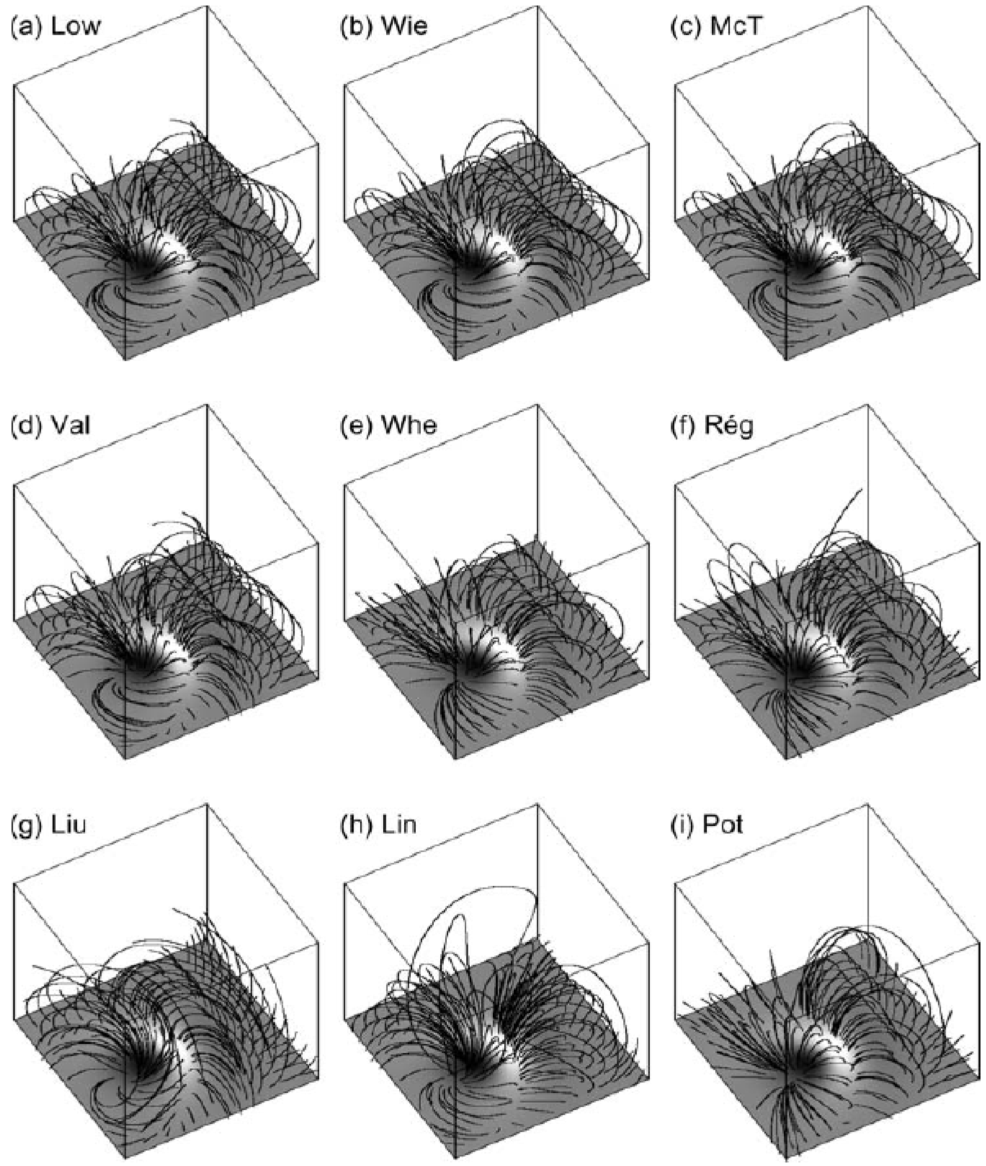
Optimization vs Grad-Rubin

- In *Schrijver et al.* (2006) and *Amari et al.* (2006), a quantitative comparison of different numerical nonlinear force-free fields was performed, based on the *Low & Lou* solutions (1990);

- The different methods performed well in strong field regions;

- The methods are compared in terms of the speed of computation, and of the accuracy with respect to the semi-analytical model;

- The authors gave a ”ranking” of methods as shown in the image on the right.



Metrics

In addition to the angle between the magnetic field and the current density, the following metrics can be derived to estimate to goodness of the extrapolations:

Vector correlation

$$C_{\mathrm{vec}} \equiv \sum_i \boldsymbol{B}_i \cdot \boldsymbol{b}_i \Bigg/ \left(\sum_i |\boldsymbol{B}_i|^2 \sum_i |\boldsymbol{b}_i|^2 \right)^{1/2},$$

Cauchy-Schwartz

$$C_{\mathrm{CS}} \equiv \frac{1}{M} \sum_i \frac{\boldsymbol{B}_i \cdot \boldsymbol{b}_i}{|\boldsymbol{B}_i||\boldsymbol{b}_i|} \equiv \frac{1}{M} \sum_i \cos\theta_i$$

Normalised vector error

$$E_{\mathrm{n}} = \sum_i |\boldsymbol{b}_i - \boldsymbol{B}_i| \Bigg/ \sum_i |\boldsymbol{B}_i|.$$

Mean vector error

$$E_{\mathrm{m}} = \frac{1}{M} \sum_i \frac{|\boldsymbol{b}_i - \boldsymbol{B}_i|}{|\boldsymbol{B}_i|}.$$

Model	C_{vec} [1]	C_{CS} [2]	E'_n [3]	E'_m [4]	ϵ [5]	ϵ_P [6]
(Case I) All boundaries provided, inner volume						
(a) Low and Lou	1	1	1	1	1	1.24
(b) Wiegelmann	1.00	1.00	0.97	0.96	1.02	1.26
(c) McTiernan	1.00	0.99	0.94	0.85	1.01	1.25
(d) Valori	1.00	0.98	0.90	0.87	0.98	1.21
(e) Wheatland	0.99	0.89	0.75	0.57	0.93	1.16
(f) Régnier	0.95	0.74	0.59	0.39	0.82	1.02
(g) Liu	0.98	0.85	0.71	0.43	0.89	1.11
(h) LinearFF	0.88	0.91	0.54	0.49	0.80	1.00
(i) Potential	0.86	0.87	0.50	0.44	0.81	1
(Case II) Only lower boundary provided, inner volume						
(a) Low and Lou	1	1	1	1	1	1.10
(b) Wiegelmann	1.00	0.91	0.92	0.66	1.04	1.14
(c) McTiernan	1.00	0.88	0.91	0.62	1.04	1.14
(d) Valori	0.99	0.82	0.83	0.39	1.02	1.12
(e) Wheatland	0.99	0.88	0.77	0.57	0.96	1.05
(f) Régnier	0.94	0.80	0.63	0.43	0.74	0.82
(g) Liu	0.97	0.54	0.48	−2.2	0.99	1.09
(h) LinearFF	0.94	0.53	0.39	−3.1	1.01	1.10
(i) Potential	0.92	0.66	0.57	0.30	0.91	1

Magnetohydrostatic magnetic fields

1. ***Equation and boundary conditions***
2. ***Analytical solutions***

Equations

Magneto-hydrostatic equilibrium

$$-\vec{\nabla} p + \rho \vec{g} + \vec{j} \wedge \vec{B} = \vec{0}$$

As for force-free, force and torque balance equations can be derived (Aly 1989) and the virial theorem:

$$\int \left(3\mathrm{p} + \frac{\mathrm{B}^2}{2\mu_0} \right) \mathrm{dV} = \int \left[(\mathrm{p} + \frac{\mathrm{B}^2}{2\mu_0})(\vec{\mathrm{n}}.\overleftarrow{\mathrm{r}}) - \frac{\mathrm{B}^2}{2\mu_0} (\vec{\mathrm{r}}.\vec{\mathrm{b}})(\vec{\mathrm{n}}.\vec{\mathrm{b}}) \right] \mathrm{dS}$$

Analytical solutions

Low et al. (1982, 1984, 1985, 1991, 1992, 1993), Neukirch (1995) developed magnetohydrostatic codes with gravitational potential.

Low (1992): ***lmhs***

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} + f(z) \nabla B_z \times \mathbf{u}_z$$

$$p = p_0(z) - \delta p = p_0(z) - f(z) \frac{B_z^2}{2\mu_0} ,$$

$$\rho = \rho_0(z) - \delta \rho = -\frac{1}{g} \frac{\mathrm{d} p_0}{\mathrm{d} z} + \frac{1}{\mu_0 g} \left[\frac{1}{2} \frac{\mathrm{d} f}{\mathrm{d} z} B_z^2 + f (\mathbf{B} \cdot \nabla) B_z \right]$$

where $f(z) = a \ \exp(-z/H)$

END