Magnetospheric Waves - A Numerical Study

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Overview

1. Introduction
2. Model
3. Code
4. Test Simulations
   - Initial Disturbance in $u_x$
   - Driving the Magnetopause Boundary with $u_x$
   - Driving the Magnetopause Boundary with $b_z$
5. Results
   - Observational Motivation and Simulation
6. Summary
Magnetospheric Structure

Solar Wind
Van Allen Radiation Belts
Tail Lobe
Plasma Sheet
Plasmasphere
Magnetosheath
Magnetopause
Plasma Mantle
Cusp
Bow Shock
Southward IMF
Southward IMF

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What are Magnetic Pulsations?

- Low frequency oscillations in the Earth’s magnetic field, detected by ground and space magnetometers.
- Named Ultra Low Frequency (ULF) waves - 1 mHz to 1 Hz.
- Quasi-sinusoidal waveforms denoted by Pc for pulsations continuous.
- Then ’Pc’ is divided into 5 classes - Pc 1-5 depending on frequency.
- In this talk we look at the Pc4 category, covering periods of 45 to 150 seconds.
Where do they come from?

- **Solar wind** - originate at sun, or through variations in the solar wind density, causing dynamic pressure fluctuations.
- **Ion foreshock** - Upstreaming ions interacting with the solar wind.
- **Bow shock** - Quasi-parallel shock, waves can escape along field lines.
- Other types to do with magnetopause motions, and internally generated waves.
Wave Coupling and Resonance

- In a cold uniform plasma, Alfvén and fast waves don’t couple.
- In the magnetosphere, the nonuniformity couples these modes.
- This leads to the idea of resonance.
- When the Alfvén frequency equals the fast mode frequency, the Alfvén mode will be resonantly excited.
- Field line resonance (FLR) - toroidal (Alfvén) and poloidal (fast) modes.
- Radially adjacent field lines have different fundamental frequencies → phase mixing.
We model the magnetosphere using a waveguide based on the hydromagnetic box implemented by *Kivelson and Southwood* [1986].

- Uniform background magnetic field $\mathbf{B} = B\hat{z}$.
- Imagine straightening the dipole field lines s.t. $\pm z$ represent the northern and southern ionospheres. (Fieldline footpoints).
- $\hat{x}$ radially outwards, $\hat{y}$ azimuthal coordinate (around earth).
- Let $\rho = \rho(x)$.
MHD Equations

- Ideal low beta plasma (low pressure).
- ⇒ We drop the energy equation.

\[
\begin{align*}
\frac{\partial B}{\partial t} &= \nabla \times (u \times B), \\
\rho \frac{\partial u}{\partial t} + \rho (u \cdot \nabla) u &= \frac{1}{\mu} j \times B - \nabla p + \rho g, \\
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) &= 0.
\end{align*}
\]
Linearization

- Equilibrium is given by $\mathbf{B} = B_0 \hat{\mathbf{z}}$, $\mathbf{u}_0 = 0$, $\rho = \rho(x)$.

\[
\begin{align*}
\frac{\partial b_x}{\partial t} &= B_0 \frac{\partial u_x}{\partial z}, \\
\frac{\partial b_y}{\partial t} &= B_0 \frac{\partial u_y}{\partial z}, \\
\frac{\partial b_z}{\partial t} &= -B_0 \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right), \\
\rho \frac{\partial u_x}{\partial t} &= \left( \frac{\partial b_x}{\partial z} - \frac{\partial b_z}{\partial x} \right) \frac{B_0}{\mu}, \\
\rho \frac{\partial u_y}{\partial t} &= \left( \frac{\partial b_y}{\partial z} - \frac{\partial b_z}{\partial y} \right) \frac{B_0}{\mu}, \\
\frac{\partial u_z}{\partial t} &= 0.
\end{align*}
\]
Standing mode in $z$, allows us to prescribe the $z$ dependence. Take

$$u_x = \bar{u}_x(x, y, t) \cos(k_z z)$$

This defines the $z$ dependence of all other quantities too.

Now all variables just dependent on $x, y$ and $t$.

Normalise magnetic field by $B_0$, length by width of waveguide, density by $\rho(x)$ at $x = 0$. Characteristic speed is given by the Alfvén speed at $x = 0$, which defines a timescale as the Alfvén transit time.
The equations then reduce to

\[
\begin{align*}
\frac{\partial b_x}{\partial t} &= -k_z u_x, \\
\frac{\partial b_y}{\partial t} &= -k_z u_y, \\
\frac{\partial b_z}{\partial t} &= -\left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right), \\
\frac{\partial u_x}{\partial t} &= \frac{1}{\rho} \left( k_z b_x - \frac{\partial b_z}{\partial x} \right), \\
\frac{\partial u_y}{\partial t} &= \frac{1}{\rho} \left( k_z b_y - \frac{\partial b_z}{\partial y} \right).
\end{align*}
\]
Finite Difference Method

- System has been expressed in terms of 5 first order pdes involving velocity rather than as 3 second order pdes involving displacement $\xi$.
- We integrate the system forward in time using a leapfrog-trapezoidal algorithm [Zalesak, 1979].
- Code written from scratch.
- We must know all of the dependent variables at times $t$ and $t - \Delta t$.
- How does this work?
Finite Difference Method

Can express equations in the form

\[ \frac{\partial \mathbf{U}}{\partial t} = \mathbf{F} \]

where

\[ \mathbf{U} = \begin{pmatrix} u_x \\ u_y \\ b_x \\ b_y \\ b_z \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \frac{(k_z b_x - b_z, x)}{\rho} \\ \frac{(k_z b_y - b_z, y)}{\rho} \\ -k_z u_x \\ -k_z u_y \\ - (u_x, x + u_y, y) \end{pmatrix} \]
Finite Difference Method

- Assuming we know $U$ at times $t$ and $t - \Delta t$, then the scheme is

\[
\begin{align*}
U^\dagger &= U^{t-\Delta t} + 2\Delta t F^t \\
F^* &= \frac{1}{2} \left( F^t + F^\dagger \right), \\
U^{t+\Delta t} &= U^t + \Delta t F^*. 
\end{align*}
\]

- Use centered finite differences to calculate the spatial derivatives.
- Scheme is second order accurate in time and space.
- May require another algorithm to get started if 2 timesteps are not known.
Boundary and Initial Conditions

- Choose an initial profile for \( u_x \):
  \[
  u_x = \sin(\pi x) \cos \left( \frac{\pi}{2} y \right)
  \]
- Box dimensions: \( x : 0 \rightarrow 1, y : 0 \rightarrow 2 \), 100 points in each.
- Perfectly reflecting boundaries at \( x = 0 \) and \( x = 1 \).
- Perfectly reflecting at \( y = 0 \) and \( y = 2 \).
- Use an FTCS method to start the time stepping.
- Wavenumbers chosen initially as \( k_x = \pi, k_y = \frac{\pi}{2} \) with \( k_z = \pi \) fixed.
- Uniform density \( \rho = 1 \).
Initial $u_x \Rightarrow$ can choose an initial $u_y$ profile s.t. $\frac{\partial b_z}{\partial t} = 0$,  

\[ u_y = -2\cos(\pi x)\sin\left(\frac{\pi}{2} y\right) \]

- Corresponds to the Alfvén mode, since there’s no magnetic pressure perturbation with $b_z = 0$.
- Can predict frequencies using the Alfvén dispersion relation.

\[ \omega = k_z v_A \]
Time sequence of $u_x$

Figure: Time sequence of $u_x$ for the Alfvén mode

$$\tau = \frac{2\pi}{\omega} = \frac{2\pi}{k_z v_A} = \frac{2\pi}{\pi} = 2.$$
Initial Disturbance in $u_x$
Driving the Magnetopause Boundary with $u_x$
Driving the Magnetopause Boundary with $b_z$

$u_x$ evolution
Fast Mode

Can introduce the fast mode here by making sure \( \frac{\partial b_z}{\partial t} \neq 0 \) by our choice of an initial profile in \( u_y \).

- Fast mode disp relation for cold plasma

\[ \Rightarrow \omega^2 = k^2 v_A^2. \]

- \( u_y = \frac{1}{2} \cos(\pi x)\sin \left( \frac{\pi}{2} y \right) \).

- \( \omega^2 = (k_x^2 + k_y^2 + k_z^2) \Rightarrow \omega = \frac{3\pi}{2} \)

- \( \tau = \frac{4}{3} \)

Figure: \( u_x \) - fast mode
Energy Conservation

Can also show conservation of energy within the code, since we start off with an initial profile with perfectly reflecting boundaries we don’t add any energy into the system.
Rather than having some initial disturbance, we set all variables to 0 and drive the outer boundary [Rickard & Wright, 1994].

Want to model exterior sources of ULF waves, perhaps caused by increased solar dynamic pressure.

Driver form with standing mode in \( y \) given by

\[
  u_x(x = 1) = \begin{cases} 
  \cos(k_y y) \left( \sin^2 \left( \frac{\pi}{\tau_D} t \right) \right) & \text{if } 0 \leq t \leq \frac{\tau_D}{2} \\
  \cos(k_y y) \left( \cos \left( 2\pi \left( \frac{t}{\tau_D} - \frac{1}{2} \right) \right) \right) & \text{if } \frac{\tau_D}{2} < t \leq T 
  \end{cases}
\]

where \( \tau_D \) is the driver period.
Calculating Expected Frequencies

- We want to know the dominant natural frequencies of the waveguide such that we can predict how our system will respond to certain driving frequencies.
- Considering our guide of length 1 in $x$, together with the boundary conditions for $u_x$, the harmonics look like:
  
  \[
  \frac{n\lambda}{2} = 1 \Rightarrow k_x = \pi n.
  \]

  \[
  \omega_n^2 = \pi^2 \left(n^2 + \frac{1}{4} + 1\right)
  \Rightarrow \tau_n = \frac{2\pi}{\omega_n}.
  \]
Calculating Expected Frequencies

- So first and second harmonics given by $\tau_1 = 1.333$ and $\tau_2 = 0.873$.
- Drive the system at $\tau_D = 1.15$, in between 2 fast frequencies.
- FFT plots clearly show the 2 fast freqs $\omega_1 = 4.712$, $\omega_2 = 7.198$ and driv freq $\omega_D = 5.46$ would be more prominent if driven for longer.
Initial Disturbance in $u_x$
Driving the Magnetopause Boundary with $u_x$

Driving the Magnetopause Boundary with $b_z$
Driving in $b_z$

- Why drive with $b_z$ rather than $u_x$?
- FFT show for driven boundary in $u_x$, still have cavity mode frequencies, acts like a perfectly reflecting boundary.
- Mann *et al.*[1999] found that cavity modes adopt closer to an antinodal BC in $u_x$.
- Antinode of $u_x$ means a node of $b_z$.
- Driving in $u_x$ simulates a node of $u_x \Rightarrow$ drive in $b_z$ to simulate antinode of $u_x$.
- This is where we enter new territory...
Harmonics and Frequencies

- Harmonic frequencies given as $\omega_1 = 3.85$, $\omega_2 = 5.88$ and $\omega_3 = 8.60$.
- Driving frequency of $\omega_D = 2\pi$.
- See 3 clear peaks.
Non-Uniform Density

As in *Wright & Rickard* [1995], we choose an Alfvén speed variation of

\[ V(x) = 1 - \frac{x}{x_0}, \quad 0 < x < x_c \]

\[ \frac{1}{V^2(x)} = \frac{(x_0 - 2x_c + 1)(1 - x_c) - (1 - x)^2}{x_0 \left(1 - \frac{x_c}{x_0}\right)^3 (1 - x_c)}, \quad x_c < x < 1. \]

- \( x_0 \) can be varied from 0.95 to 1.8.
  \( \Rightarrow \) How fast \( V \) changes with L shell.
- \( x_c \) is the point where the form of \( V_A \) changes.
- Have \( V \) and \( \frac{dV}{dx} \) continuous at \( x_c \).
- \( \frac{dV}{dx} = 0 \) at \( x = 1 \).
Phase Mixing Length

- Define the phase mixing length in $x$ as the length over which the phase of neighbouring Alfvén waves differ by $2\pi$.
- $L_{ph} \sim 2\pi / (td\omega_A/dx)$ Mann et al.[1995].
- Phase mixing length must always be resolved.
- $L_{ph} \sim \frac{1}{t}$, so longer simulations require finer scales in $x$.
- Choice of Alfvén speed profile gives a phase mixing length independent of $x$. 
Paper by Clausen et al. 2008, observed a large scale Pc4 pulsation, seen at almost all latitudes.

Caused by an external source (backstreaming ions into the solar wind).

Results from both space and ground based data.

They present evidence that the driving wave interacted resonantly with magnetic fieldlines.

Suggest that resonant driving occurs at stations where the driving freq was harmonically related to the local fundamental frequency creating FLR like structures.
Our Aims

- Want to display some of the resonance results discussed in Clausen et al. [2008].
- Does our code produce a field line resonance given a specific driving frequency?
- Is driving in $b_z$ an effective mechanism to mimic external wave sources?
Set up

- Normalize to outer ($x = 1$) boundary instead of the inner boundary.
- From Clausen et al. [2008], we have an estimate of fundamental field line frequency variation with L-shell (radius).
Use Alfvén speed at $x=1$ to normalize by, this will give a normalizing time too, using a normalizing length of $10_{RE}$.

Lowest frequency at $10_{RE} \sim 0.006\,Hz \Rightarrow$ normalizing $V_A = 764.5\,kms^{-1}$, $\tau_A = 83.33\,s$.

Need to change the value of $x_0$ in the Alfvén speed profile, such that fundamental freqs match as predicted by Clausen et al. [2008].

Driving frequency is chosen as $17.2\,mHz$, which is the frequency of the pulsation in the paper.
Clausen et al. main frequency
Want to increase length of time of simulation, recall $L_{ph} \sim 1/t$.

Have to increase the grid resolution in $x$.

This implies, by the CFL condition, that the timestep $\Delta t$ is reduced.

Also want no reflection off the inner $y$ boundary $\Rightarrow$ increase the length of the guide $y : 0 \rightarrow 10$

Just drive the $y : 0 \rightarrow 1$ portion of the waveguide.
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This implies, by the CFL condition, that the timestep $\Delta t$ is reduced.

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Just drive the $y : 0 \rightarrow 1$ portion of the waveguide.

This takes just 4 minutes on my machine!
Alfvén Energy Variation
We can predict the point in $x$ where the resonance will occur by looking at the Alfvén disp. relation.

$$\omega(x) = k_z v_A(x)$$

Predicted resonance at $x = 0.41$
Code does show similar evidence of downtail propagation of energy, as in *Clausen et al.*, [2008].

Taken downtail away from driving region $\Rightarrow$ Satellite must have been fairly anti-sunward to get this signature.
Poynting Flux

- Taken at a point within the driving range in $y$.
- Clearly not just tailward propagation.
Attempting to model ULF waves in the magnetosphere.

We have written and tested a code for solving the cold plasma equations using the box model of Kivelson & Southwood [1986].

We tried a new driving condition so we drive in $b_z$ rather than $u_x$.

We find that our code can produce FLR type behaviour.

Still looking at meaning of Poynting flux.