An Introduction to Waves in Earth’s Magnetosphere

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Overview

- Introduction to the Magnetosphere.
- Ultra Low Frequency (ULF) Waves.
- What we do with these waves.
Have an appreciation for the overall structure of the magnetosphere.

ULF Waves - theory, generation, observation.

Modeling - how to model ULF waves in the magnetosphere.
So now you know…

- Bow Shock
- Magnetosheath
- Magnetopause
- Van Allen Belts
- Plasmasphere/pause
- Magnetotail
- Low frequency oscillations in Earth’s magnetic field.
- First discovered in the mid 1800s using a microscope and a very long compass needle.
- Typical magnetic oscillations of around a few $nT$.
- ULF waves cover frequencies from $1\ mHz$ to $1\ Hz$.
- Importance: can drive currents in the ionosphere.
- Understanding the dynamics of the magnetosphere.
Where do they come from?

- **Solar wind** - originate at sun, or through variations in the solar wind density, causing dynamic pressure fluctuations.

- **Kelvin-Helmholtz Instability** - Heightened solar wind flow speeds lead to increased magnetosheath flow and the magnetopause can become KH unstable on the flanks.

- Other types e.g. internally generated waves.
The outer magnetosphere can be thought of as a natural waveguide. The magnetopause provides an outer boundary, with the plasmapause as a possible internal boundary.

Samson et al. [1992] found that similar discrete frequencies kept occurring in the outer magnetosphere. This developed the idea of the cavity selecting the frequencies based on its size and shape, just like an instrument.

Wright and Rickard [1995] showed discrete frequencies excited by a broadband frequency driver.
In a cold uniform plasma, Alfvén and fast waves don’t couple.
In the magnetosphere, the nonuniformity couples these modes.
Where the Alfvén frequency equals the fast mode frequency, can excite a field line resonance (FLR).
This has detectable observational features.
We model the magnetosphere using a waveguide based on the hydromagnetic box implemented by *Kivelson and Southwood* [1986].

**Uniform background magnetic field** $B = B \hat{z}$, picture straightening dipole fieldlines.

* $\hat{x}$ radially outwards, $\hat{y}$ azimuthal coordinate (around earth).

Let $\rho = \rho(x)$. 

Wright & Rickard 1995
Ideal low beta plasma (low pressure). (Cold plasma equations)

⇒ Energy equation is zero.

\[
\begin{align*}
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{u} \times \mathbf{B}), \\
\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} &= \frac{1}{\mu} \mathbf{j} \times \mathbf{B} - \nabla \rho + \rho \mathbf{g}, \\
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0.
\end{align*}
\]
Linearise and assume perturbations as $b_z(x)e^{i(k_y y + k_z z - \omega t)}$

$$\frac{d^2 b_z}{dx^2} - \frac{\omega^2 dV_A}{\omega^2/V_A^2 - k_z^2} \cdot \frac{db_z}{dx} + \left( \frac{\omega^2}{V_A^2} - k_y^2 - k_z^2 \right) b_z = 0$$

Second order ordinary differential equation for $b_z(x)$, the $z$ component of the magnetic field.

Have assumed a density variation only in $x$.

Singularity when $\omega^2 = V_A^2 k_z^2$, at a specific location in $x$.

Represents location where energy from longitudinal waves is converted into transverse waves, like waves along a string.
We’re interested in numerical modeling so express as

\[
\begin{align*}
\frac{\partial b_x}{\partial t} &= -k_z u_x, \\
\frac{\partial b_y}{\partial t} &= -k_z u_y, \\
\frac{\partial b_z}{\partial t} &= -\left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right), \\
\frac{\partial u_x}{\partial t} &= \frac{1}{\rho} \left( k_z b_x - \frac{\partial b_z}{\partial x} \right), \\
\frac{\partial u_y}{\partial t} &= \frac{1}{\rho} \left( k_z b_y - \frac{\partial b_z}{\partial y} \right).
\end{align*}
\]

which we solve using a Leapfrog-Trapezoidal finite difference scheme. [Zalesak, 1979]
Waveguide & Boundary Conditions

- Waveguide of length 1 in x, and 10 in y. 1 unit = 10\(R_e\).
- Perfectly reflecting boundary at \(x = 0\) (plasmapause).
- Driven boundary at \(x = 1\) (magnetopause).
- Symmetry condition at \(y = 0\) \(\Rightarrow u_y = 0\).
- Open ended in \(y\) \(\rightarrow\) wave never reaches end of box in \(y\).
Driving with $b_z$.

- Drive with $b_z$ perturbation, rather than displacement as previously, to mimic pressure driving.
- Changes the boundary condition at the magnetopause: closer to a node of $b_z$ than $u_x$.
- Can provide a more realistic range of waveguide eigenfrequencies (natural guide frequencies) [Mann et al., 1999].

Figure shows radial waveguide harmonics for a uniform density medium.
Applying the Model - Aims

- Take an observation from a paper.
- Attempt to model the equilibrium from the given parameters.
- Match satellite positions.
- Ask ourselves: Can we match to their results?
- Can we question their conclusions?
- What new information/explanation do we have?
Applying the Model - The Observation

- Observation from THEMIS
- Frequency 6.5mHz global mode.
- Dominant $b_z$ and $E_y$ perturbations.
- Radially inwards $S_x$.
- THD location: magnetic latitude $\sim 3^\circ$, near plasmapause, radially aligned with source region.

[Hartinger et al., 2012]
Applying the Model - THEMIS Results

[Images of graphs and plots showing time series data for $b_z$, $b_y$, $b_x$, $u_y \sim E_x$, $u_x \sim E_y$, $S_z$, $S_y$, and $S_x$ with annotations indicating the THEMIS experiment results.

[Elsden and Wright, 2015]
Applying the Model: Phase Shifts

- Difference between driving and post driving phase shifts.
- Can be used in observations to infer the end of the driving phase.
Wanted to analyse the phase shift together with changes in $S_x$. 

\[ S_x = u_x b_z. \]

We're a theoretical group, so time for some theory! Consider an inward propagating wave $+$ smaller amplitude reflected wave. 

\[ u_x = \cos(\omega t + k_x x - k_y y) \cos(k_z z) + R \cos(\omega t - k_x x - k_y y) \cos(k_z z). \]
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\[ + R \cos(\omega t - k_x x - k_y y) \cos(k_z z). \]
Let $k_y = 0$, as an approximation for a global mode.

Close to magnetic equator at $z = 0$. 

$u_x = \cos(\omega t + k_x x) + R \cos(\omega t - k_x x)$,

$b_z = -A' \cos(\omega t + k_x x) + A' R \cos(\omega t - k_x x)$,

for $A' = k_x / \omega$. 

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Let $k_y = 0$, as an approximation for a global mode.

Close to magnetic equator at $z = 0$.

Substitute form for $u_x$ into equations.

$$u_x = \cos(\omega t + k_x x) + R \cos(\omega t - k_x x),$$

$$b_z = -A' \cos(\omega t + k_x x) + A'R \cos(\omega t - k_x x),$$

for $A' = k_x/\omega$. 
Try to express components in terms of a single sinusoid.

\[ u_x = G \cos(\omega t + \psi), \]
\[ b_z = G' \cos(\omega t + \psi'), \]

\( G, G', \psi, \psi' \) all functions of \( k_x x \) and \( R \).
Applying the Model: Phase Shifts

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Then we can define the phase shift as

\[ \phi = \pi + \tan^{-1} \left( \frac{2R}{1 - R^2} \sin(2k_xx) \right). \]
Now to compare this to the Poynting vector.

\[ S_x = u_x b_z, \]
\[ = -A' \cos^2(\omega t + k_x x) + R^2 A' \cos^2(\omega t - k_x x). \]
Applying the Model: Phase Shifts

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\[ = -A' \cos^2(\omega t + k_x x) + R^2 A' \cos^2(\omega t - k_x x). \]

Again express as a single sinusoid.

\[ S_x = \gamma + C \sin(2\omega t + \delta), \]
for \( \gamma, C \) and \( \delta \) all functions of \( k_x x \) and \( R \).
Applying the Model: Phase Shifts

Consider the ratio of positive to negative Poynting vector signal, defining the 'shape'.

\[
\Delta_s = \left| \frac{R^2 - 1 + \sqrt{R^4 + 1 - 2R^2 \cos(4k_x x)}}{R^2 - 1 - \sqrt{R^4 + 1 - 2R^2 \cos(4k_x x)}} \right|.
\]
Applying the Model: Phase Shifts

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So we have expressions for the phase shift between \( u_x \) and \( b_z \), and the 'shape' of the radial Poynting vector. How are they related?
Well it turns out they have the same contours in \((R, k_x x)\) space. Awesome right!?
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This means that one can be expressed as a function of the other and hence shows how these two quantities are linked.
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This means that one can be expressed as a function of the other and hence shows how these two quantities are linked.

So can think of the phase difference determining the shape of the Poynting vector and vice versa.
\begin{align*}
\phi & := \text{arctan}\left(\frac{2 \cdot y}{1 - y^2} \cdot \sin(2 \cdot x)\right); \\
\delta & := \frac{1 + \sqrt{y^4 + 1 - 2 \cdot y^2 \cdot \cos(4 \cdot x)}}{y^2 - 1 - \sqrt{y^4 + 1 - 2 \cdot y^2 \cdot \cos(4 \cdot x)}}; \\
\phi & := \text{arctan}\left(\frac{2 \cdot y \cdot \sin(2 \cdot x)}{-y^2 + 1}\right);
\end{align*}
\[ > \text{diff}(\phi, x) \cdot \text{diff}(\delta, y) - \text{diff}(\phi, y) \cdot \text{diff}(\delta, x); \]

\[
\begin{align*}
4y\cos(2x) & \left( \frac{2y + \frac{1}{2} \frac{4y^3 - 4y\cos(4x)}{\sqrt{y^4 + 1 - 2y^2\cos(4x)}}}{y^2 - 1 - \sqrt{y^4 + 1 - 2y^2\cos(4x)}} \right) \\
& - \frac{1}{1 + \frac{4y^2\sin(2x)^2}{(-y^2 + 1)^2}} \left( \left( \frac{2\sin(2x)}{-y^2 + 1} + \frac{4y^2\sin(2x)}{(-y^2 + 1)^2} \right) \left( \frac{4y^2\sin(4x)}{\sqrt{y^4 + 1 - 2y^2\cos(4x)} \left( y^2 - 1 - \sqrt{y^4 + 1 - 2y^2\cos(4x)} \right)^2} \right) \right)
\end{align*}
\]
Maple check

\[ \texttt{simplify}(\%) = 0 \]
The driven and undriven phases can be distinguished either through $S_x$ shape or the phase shift of $u_x$ and $b_z$. What was the point?
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Know what phase shift / radial Poynting vector ratios are expected for global modes ⇒ useful for global mode identification in data.
The driven and undriven phases can be distinguished either through $S_x$ shape or the phase shift of $u_x$ and $b_z$.

Know what phase shift / radial Poynting vector ratios are expected for global modes $\Rightarrow$ useful for global mode identification in data.

Contour plots of the 2 quantities can constrain values of reflection $R$ and $k_x x$. E.g. if phase shift $= 120^\circ$ then $0.6 < R < 1$. 
What have we learned from the model?

- Confirmation of global mode signature.
- Despite model simplifications still a very good match to the data.
- Correlated phase shifts with Poynting vector signals.
- Can infer satellite position in reference to the energy source location through the Poynting vector.
- Displays the power of even very simplified models in revealing information not immediately apparent from the data.
Conclusions

- So I said I’d like you to have learned something...
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- Magnetospheric structure
- ULF waves: generation, theory, observations.
- Wave modeling...
Conclusions

- So I said I’d like you to have learned something...
- Magnetospheric structure
- ULF waves: generation, theory, observations.
- Wave modeling...
- ...and hopefully you had a good sleep for the last part!
Thanks!

An Introduction to Waves in Earth's Magnetosphere
Can express equations in the form

\[ \frac{\partial \mathbf{U}}{\partial t} = \mathbf{F} \]

where

\[ \mathbf{U} = \begin{pmatrix} u_x \\ u_y \\ b_x \\ b_y \\ b_z \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} (k_z b_x - b_z, x) / \rho \\ (k_z b_y - b_z, y) / \rho \\ -k_z u_x \\ -k_z u_y \\ -(u_x, x + u_y, y) \end{pmatrix} \]
Assuming we know $U$ at times $t$ and $t - \Delta t$, then the scheme is

$$U^\dagger = U^{t-\Delta t} + 2\Delta t F^t$$

$$F^* = \frac{1}{2} \left( F^t + F^\dagger \right),$$

$$U^{t+\Delta t} = U^t + \Delta t F^*. $$

- Use centered finite differences to calculate the spatial derivatives.
- Scheme is second order accurate in time and space.


