Magnetic Nullpoints in 2 dimensions

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A magnetic null point is a point in a magnetic field where all the components of the field are zero. In 2 dimensions:

$$B_x = B_y = 0$$

This tells us little in itself about the local magnetic structure; the topology of the field in the immediate vacinity of one null point may be quite different near another. However, if we assume that the magnetic field near a null point approaches zero linearly, we can approximate the components of the magnetic field in this region by means of a two variable, first order Taylor expansion about the neutral point X_0, Y_0. Consider the x component:

$$B_X \approx B_X(X_0, Y_0) + \frac{\partial B_X}{\partial X}_{X_0,Y_0}(X - X_0) + \frac{\partial B_X}{\partial Y}_{X_0,Y_0}(Y - Y_0)$$

Retain only the first order, linear terms:

$$B_X \approx \frac{\partial B_X}{\partial X}_{X_0,Y_0}(X - X_0) + \frac{\partial B_X}{\partial Y}_{X_0,Y_0}(Y - Y_0)$$

Choose an origin such that $X_0 = Y_0 = 0$.

$$B_X \approx \frac{\partial B_X}{\partial X}_{0,0} X + \frac{\partial B_X}{\partial Y}_{0,0} Y$$

Similarly for y:

$$B_Y \approx \frac{\partial B_Y}{\partial X}_{0,0} X + \frac{\partial B_Y}{\partial Y}_{0,0} Y$$

We may then express the magnetic field near a null point (to lowest order) as

$$\mathbf{B} = \mathbf{M}.\mathbf{r}$$

where $M = \begin{pmatrix} \frac{\partial B_X}{\partial X} & \frac{\partial B_X}{\partial Y} \\ \frac{\partial B_Y}{\partial X} & \frac{\partial B_Y}{\partial Y} \end{pmatrix}$ and $r = (X, Y)^T$

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[*] In this section I will be drawing heavily on the theory developed by Parnell et al. in *The Structure of Three-Dimensional Magnetic Neutral Points*, but justifying the results stated in this paper in greater mathematical detail. It is possible that, at some points, Parnell may have a different approach in mind to the one I have adopted.

For simplicity, we rewrite the above matrix as

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

However, this matrix can be simplified and rewritten in a form that will lend itself more readily to meaningful analysis. First, we impose the solenoidal constraint,

$$\nabla.\mathbf{B} = 0 \Rightarrow \frac{\partial B_X}{\partial X} = \frac{\partial B_Y}{\partial Y} = 0 \Rightarrow a_{11} = a_{22} = 0. \Rightarrow a_{11} = -a_{22}$$

The diagonal entries are associated with the potential part of the field (they do not show up in the expression for the current below), so we let $a_{11} = p$, $a_{22} = -p$.

Consider now the current,

$$\mathbf{J} = \frac{1}{\mu_0}\nabla \times \mathbf{B} = \frac{1}{\mu_0}\nabla \times \left(\begin{array}{cc} p & a_{12} \\ a_{21} & -p \end{array} \right) = \frac{1}{\mu_0}(0, 0, a_{21} - a_{12})$$

We can conveniently rewrite

$$a_{12} = \frac{1}{2}(q - J_z),\ a_{21} = \frac{1}{2}(q + J_z)$$

Thus for a current-free neutral point, where $J_z = 0$, $a_{12} = a_{21} = \frac{q}{2}$. Therefore the parameter q is associated with the potential field. The matrix $\mathbf{M}$ may now be stated in its final form:

$$\mathbf{M} = \left(\begin{array}{cc} p & \frac{1}{2}(q - J_z) \\ \frac{1}{2}(q + J_z) & -p \end{array} \right) \tag{1}$$

1 The Threshold Current

From the square root of the discriminant of the characteristic equation of the symmetric part of $\mathbf{M}$, we define a threshold current,

$$J_{thresh} = \sqrt{4p^2 + q^2} \tag{2}$$

which we note is only dependent on parameters associated with the potential part of the field. The proof proceeds as follows:

$$M_S = \frac{1}{2}(M + M^T)$$

$$= \frac{1}{2}\left[\left(\begin{array}{cc} p & \frac{1}{2}(q - J_z) \\ \frac{1}{2}(q + J_z) & -p \end{array} \right) + \left(\begin{array}{cc} p & \frac{1}{2}(q + J_z) \\ \frac{1}{2}(q - J_z) & -p \end{array} \right) \right]$$

$$= \frac{1}{2}\left(\begin{array}{cc} p & \frac{1}{2}q \\ \frac{1}{2}q & -p \end{array} \right)$$

$$det(M_S - \lambda) = -p^2 + \lambda^2 - \frac{q^2}{4} = 0 \Rightarrow \lambda^2 - (p^2 + \frac{q^2}{4}) = 0$$

This yields a discriminant

$$d = 4p^2 + q^2$$

Thence:

$$J_{thresh} = \sqrt{d} = \sqrt{4p^2 + q^2}$$

as given above.

2 The Flux Function

We now determine the flux function A – an expression that characterises the geometry of the magnetic field, defined to obey the solenoidal constraint). It satisfies

$$B_X = \frac{\partial A}{\partial Y},\ B_Y = -\frac{\partial A}{\partial X}$$

Since $\mathbf{B} = \mathbf{M}.\mathbf{r}$, $B_X = pX + \frac{1}{2}(q - J_z)$, and $B_Y = \frac{1}{2}(q + J_z) - pY$. Hence:

$$A = \int B_X dY = pXY + \frac{1}{4}(q - J_z)Y^2 + f(X)$$

$$A = -\int B_Y dY = -\left(\frac{1}{4}(q + J_z)X^2 - pXY\right) + f(Y)$$

Therefore,

$$A = \frac{1}{4}\left((q - J_z)Y^2 - (q + J_z)^2 X^2\right) + pXY$$

This expression can be further simplified by a rotation of the XY axes, allowing us eventually to rewrite it as

$$A = \frac{1}{4}\left[(J_{thresh} - J_z)y^2 - (J_{thresh} + J_z)x^2\right], \tag{3}$$

ie. a function of the two parameters J_{thresh} and J_z. The proof proceeds as follows:

Rotate XY-axes through an angle θ,

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

to give

$$A = \frac{1}{4}\left((q - J_z)(xsin\theta + ycos\theta)^2 - (q + J_z)^2(xcos\theta - ysin\theta)^2\right) +$$

$$p(xcos\theta - ysin\theta)(xsin\theta + ycos\theta)$$

Expanding, and factorising in x^2, y^2 and xy, yields

$$A = x^2\left[\frac{1}{4}(q - J_z)sin^2\theta - \frac{1}{4}(q + J_z)cos^2\theta + psin\theta cos\theta\right] +$$

$$y^2\left[\frac{1}{4}(q-J_z)cos^2\theta-\frac{1}{4}(q+J_z)sin^2\theta-psin\theta cos\theta\right]+$$
$$xy\left[qsin\theta cos\theta+p(cos^2\theta-sin^2\theta)\right]$$

Now let $tan2\theta=-2\frac{p}{q}$. First, consider the xy term:

$$qsin\theta cos\theta+p(cos^2\theta-sin^2\theta)=\frac{q}{2}sin2\theta+pcos2\theta=-pcos2\theta+pcos2\theta$$

ie. the xy term disappears. Next, consider the x^2 term:

$$x^2\left[\frac{1}{4}(q-J_z)\frac{1}{2}(1-cos2\theta)-\frac{1}{4}(q+J_z)\frac{1}{2}(1+cos2\theta)+\frac{1}{2}psin2\theta\right]$$
$$=-\frac{x^2}{4}\left(qcos2\theta-2psin2\theta+J_z\right)=-\frac{x^2}{4}\left(qcos2\theta+\frac{4p^2}{q}cos2\theta+J_z\right)$$
$$=-\frac{x^2}{2}\left(\frac{1}{(4p^2+q^2)^{\frac{1}{2}}}(4p^2+q^2)+J_z\right)$$
$$=-\frac{x^2}{2}\left(J_{thresh}+J_z\right)$$

Finally, consider the y^2 term:

$$\frac{y^2}{8}\left[(q-J_z)(1+cos2\theta)-(q+J_z)(1-cos2\theta)-4psin2\theta)\right]$$
$$=\frac{y^2}{8}\left(qcos2\theta-2psin2\theta-J_z\right)=\frac{y^2}{8}\left(qcos2\theta+\frac{4p^2}{q}cos2\theta-J_z\right)$$
$$=\frac{y^2}{4}\left(\frac{1}{(4p^2+q^2)^{\frac{1}{2}}}(4p^2+q^2)-J_z\right)$$
$$=\frac{y^2}{4}\left(J_{thresh}-J_z\right)$$

And hence,

$$A=\frac{1}{4}\left[(J_{thresh}-J_z)y^2-(J_{thresh}+J_z)x^2\right]$$

3 The Eigenvalues

Finally, we determine a general expression for the eigenvalues of **M**.

$$det(M-\lambda)=(p-\lambda)(-p-\lambda)-\frac{1}{4}(q-J_z)(q+J_z)$$

$$=\lambda^2-\frac{1}{4}(4p^2+q^2-J_z^2)=\lambda^2-\frac{1}{4}(J_{thresh}^2-J_z^2)$$

Hence,

$$\lambda=\pm\frac{1}{2}\sqrt{J_{thresh}^2-J_z^2} \tag{4}$$

It is apparent that, if $J_z<J_{thresh}$, then $\lambda\in\Re$, and if $J_z>J_{thresh}$, then $\lambda\in\Im$.

4 Classifying 2d Nullpoints

4.1 Potential Null Points

A *potential* field is 'current free', ie. $J = 0 \Rightarrow J_z = 0$. It follows that $\mathbf{M}$ is symmetric in the potential case, and from (4) the eigenvalues are given by

$$\lambda = \pm\frac{1}{2}J_{thresh};$$

Thus we have two real non-zero eigenvalues. From (3) it is apparent that the flux function is simply

$$A = \frac{J_{thresh}}{4}(y^2 - x^2)$$

From the flux function we can quickly discover the geometry of the field lines.

$$B_x = \frac{\partial A}{\partial y} = \frac{J_{thresh}}{2}y,\ B_y = -\frac{\partial A}{\partial x} = \frac{J_{thresh}}{2}x.$$

$$\frac{dy}{dx} = \frac{B_y}{B_x} = \frac{y}{x} \rightarrow ydx = xdy \rightarrow y^2 - x^2 = c$$

Plainly, for $c = 0 \rightarrow y = \pm x$, and for $c \neq 0 \rightarrow y = \pm\sqrt{c + x^2}$.

The field lines are thus rectanglar hyperbola with separatrices that intersect at an angle of $\frac{\pi}{2}$. We call this an *X-type neutral point*, and it is the only possible configuration in 2d for a neutral point in a potential field.

4.2 Non-Potential Null points

Although we are concerned with a potential field extrapolation in this investigation, we briefly consider here the case of 2d neutral points in a non-potential field. 2d neutral points with current are classified by the magnitude of J_z and J_{thresh}.

1. $|J_z| < J_{thresh}$: Here, (4) → the eigenvalues are real, equal in magnitude, and opposite in sign, and (3) →A = ay$^2 - bx^2$ where $a, b > 0$, ie. the field lines are hyperbolae with separatrices that intersect at an angle of

$$tan^{-1}\left(\frac{(J_{thresh}^2 - J_z^2)^{\frac{1}{2}}}{J_z}\right)$$

Again, we have an X-type neutral point, which reduces to the potential case (rectangular hyperbolae) as $J_z \rightarrow 0$.

2. $|J_z| = J_{thresh}$: (4) → the eigenvalues are both zero. (3) →

$$A = \begin{cases} -\frac{1}{2}J_{thresh}x^2, & J_{thresh} = J_z \\ \frac{1}{2}J_{thresh}y^2, & J_{thresh} = -Jz \end{cases}$$

Hence,

$$B_x = \frac{\partial A}{\partial y} = \begin{cases} 0, & J_{thresh} = J_z \\ J_{thresh}y, & J_{thresh} = -Jz \end{cases}$$

$$B_y = -\frac{\partial A}{\partial x} = \begin{cases} J_{thresh} x, & J_{thresh} = J_z \\ 0, & J_{thresh} = -Jz \end{cases}$$

Therefore,

$$J = -J_z \longrightarrow \frac{dy}{dx} = 0 \longrightarrow y = const$$

$$J = J_z \longrightarrow \frac{dx}{dy} = 0 \longrightarrow x = const$$

Thus this configuration produces anti-parallel field lines with a null either along the x-axis or the y-axis.

3. $|J_z| > J_{thresh}$: Here, (4) → the eigenvalues are complex conjugates.

For the case $J_{thresh} = 0$, (3) →

$$A = -\frac{J_z}{4}(x^2 + y^2)$$

ie. the field lines are circular and centred around the origin.

For the case $J_{thresh} \neq 0$, (3) →

$$A = -\frac{1}{4}(ay^2 + bx^2)$$

(where a, b are constants, both greater than zero), ie. the field lines are concentric ellipses.