

# Magnetic Nullpoints in 3 dimensions

6th July 2008

Generalising from the 2d case, we find that the magnetic field,  $\mathbf{B}$ , near a 3d neutral point may be expressed to lowest order as

$$\mathbf{B} = \mathbf{M} \cdot \mathbf{r}$$

where  $M = \begin{pmatrix} \frac{\partial B_X}{\partial X} & \frac{\partial B_X}{\partial Y} & \frac{\partial B_X}{\partial Z} \\ \frac{\partial B_Y}{\partial X} & \frac{\partial B_Y}{\partial Y} & \frac{\partial B_Y}{\partial Z} \\ \frac{\partial B_Z}{\partial X} & \frac{\partial B_Z}{\partial Y} & \frac{\partial B_Z}{\partial Z} \end{pmatrix}$  and  $r = (X, Y, Z)^T$

For simplicity, rewrite the above matrix as

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Again, imposing the solenoidal constraint we find that  $a_{11} + a_{22} + a_{33} = 0$ . Since the diagonalised (or Jordan normalised) form has the same *trace*, it follows that

$$\lambda_1 + \lambda_2 + \lambda_3 = 0 \tag{1}$$

The associated eigenvectors are  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ .

## 1 The Geometry of a 3D Null

It is apropos at this point to briefly comment on the two basic components that make up the skeleton of any 3-dimensional neutral point: the *fan* of a null point is the surface formed by the set of field lines radiating out of or into the null point; the *spine* is formed by the two field lines that enter or leave the null point (not in the plane of the fan). The spine is directed away from the null if the field lines in the fan are directed towards it, and vice-versa. We will make reference to these features in the mathematical investigation of the field lines that follows.

---

\*In this section I will be drawing heavily on the theory developed by Parnell et al. in *The Structure of Three-Dimensional Magnetic Neutral Points*, but justifying the results stated in this paper in greater mathematical detail. It is possible that, at some points, Parnell may have a different approach in mind to the one I have adopted.

## 2 Tracing the Fieldlines round a Null

We can write a magnetic field line near the null in terms of a position vector  $\mathbf{r} = (x, y, z)^T$  dependent on an arbitrary parameter  $k$ ,

$$\frac{d\mathbf{r}(k)}{dk} = \mathbf{M} \cdot \mathbf{r}(k) = \mathbf{B} \quad (2)$$

Now let

$$\mathbf{r}(k) = \mathbf{P}\mathbf{u}(k) \quad (3)$$

where  $\mathbf{P}$  is the matrix of the eigenvectors of  $\mathbf{M}$ ,  $\mathbf{P} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ . Then (2)  $\rightarrow$

$$\frac{d\mathbf{u}(k)}{dk} = \mathbf{P}^{-1}\mathbf{M} \cdot \mathbf{P}\mathbf{u} \quad (4)$$

There are two cases to consider:

### 2.1 $\mathbf{M}$ is disagonalisable

In this case,  $\mathbf{M}$  can be diagonalised to a matrix  $\Lambda$ . (4)  $\rightarrow$

$$\frac{d\mathbf{u}(k)}{dk} = \Lambda\mathbf{u} \quad (5)$$

Hence,

$$\frac{d\mathbf{u}}{\mathbf{u}} = \Lambda dk \rightarrow \mathbf{u} = \mathbf{A} \exp(\Lambda k) \quad (6)$$

where  $A$  is also a diagonal matrix with entries  $A, B, C$  which are constant along a field line.

Now (7) and (6)  $\rightarrow$

$$\mathbf{r}(k) = Ae^{\lambda_1 k} \mathbf{x}_1 + Be^{\lambda_2 k} \mathbf{x}_2 + Ce^{\lambda_3 k} \mathbf{x}_3 \quad (7)$$

So each field line may be written in terms of the eigenvectors and eigenvalues of  $\mathbf{M}$ .

1. Consider the case where all the eigenvalues are real. (1)  $\rightarrow \lambda_1, \lambda_2 > 0, \lambda_3 < 0$ , ie. there is always one eigenvalue of opposite sign to the other two.

i. Let  $k \rightarrow -\infty$ .

This is tantamount to tracing a field line backwards, away from the neutral point. We find that

$$\mathbf{r}(k) \rightarrow Ce^{\lambda_3 k} \mathbf{x}_3$$

So all the field lines that head in towards the null are parallel to  $\mathbf{x}_3$ . It follows that  $\mathbf{x}_3$  (ie. the eigenvector associated with the eigenvalue of opposite sign to the other) defines the path of the spine, and that, for  $\lambda_1, \lambda_2 > 0, \lambda_3 < 0$ , the spine is heading towards the null.

ii. Let  $k \rightarrow \infty$ .

This is tantamount to tracing a field line forwards, away from the neutral point. We find that

$$\mathbf{r}(k) \rightarrow Ae^{\lambda_1 k} \mathbf{x}_1 + Be^{\lambda_2 k} \mathbf{x}_2$$

So all the field lines that are directed away from the null lie parallel to the plane defined by  $\mathbf{x}_1, \mathbf{x}_2$ . It follows that  $\mathbf{x}_1, \mathbf{x}_2$  (ie. the eigenvectors that share the same sign) define the plane of the fan, and that, for  $\lambda_1, \lambda_2 > 0, \lambda_3 < 0$ , the field lines in the fan are emanating radially outwards from the null.

2. Consider the case where we have two complex and one real eigenvalue,  $\lambda_1 = \eta + i\nu, \lambda_2 = \eta - i\nu, \lambda_3 = -2\eta$ , with corresponding eigenvectors  $x_1 = (\acute{\mathbf{x}}_1 + i\acute{\mathbf{x}}_2)/2, x_2 = (\acute{\mathbf{x}}_1 - i\acute{\mathbf{x}}_2)/2, \mathbf{x}_3$ . Then (7)  $\rightarrow$

$$\mathbf{r}(k) = \frac{1}{2}(A + iB)e^{(\eta+i\nu)k}(\acute{\mathbf{x}}_1 + i\acute{\mathbf{x}}_2) + \frac{1}{2}(A - iB)e^{(\eta-i\nu)k}(\acute{\mathbf{x}}_1 - i\acute{\mathbf{x}}_2) + Ce^{-2\eta k} \acute{\mathbf{x}}_3 \quad (8)$$

(8) can be conveniently rewritten as

$$\mathbf{r}(k) = e^{\eta k} R \cos(\Theta + \nu)k \acute{\mathbf{x}}_1 - e^{\eta k} R \sin(\Theta + \nu)k \acute{\mathbf{x}}_2 + Ce^{-2\eta k} \acute{\mathbf{x}}_3 \quad (9)$$

where  $A$  and  $B$  have been rewritten in terms of the constants  $R$  and  $\Theta$ . The proof proceeds as follows:

Consider the first two terms, and separate for  $\acute{\mathbf{x}}_1, \acute{\mathbf{x}}_2$ .

$$\begin{aligned} & \frac{1}{2}e^{\eta k} \left[ \left\{ (A + iB)e^{i\nu k} + (A - iB)e^{-i\nu k} \right\} \acute{\mathbf{x}}_1 + i \left\{ (A + iB)e^{i\nu k} - (A - iB)e^{-i\nu k} \right\} \acute{\mathbf{x}}_2 \right] \\ = & \frac{1}{2}e^{\eta k} \left[ \left\{ A(e^{i\nu k} + e^{-i\nu k}) + iB(e^{i\nu k} - e^{-i\nu k}) \right\} \acute{\mathbf{x}}_1 + i \left\{ A(e^{i\nu k} - e^{-i\nu k}) + iB(e^{i\nu k} + e^{-i\nu k}) \right\} \acute{\mathbf{x}}_2 \right] \\ = & e^{\eta k} \left[ (A \cos(\nu k) - B \sin(\nu k)) \acute{\mathbf{x}}_1 - (A \sin(\nu k) + B \cos(\nu k)) \acute{\mathbf{x}}_2 \right] \end{aligned}$$

Now,

$$\begin{aligned} \cos(\theta k + \nu k) &= \cos(\theta k) \cos(\nu k) - \sin(\theta k) \sin(\nu k) \\ \sin(\theta k + \nu k) &= \sin(\theta k) \cos(\nu k) + \cos(\theta k) \sin(\nu k) \end{aligned}$$

So let  $\sin(\theta k) = B/R, \cos(\theta k) = A/R$ . Hence we can rewrite the above, including the third term, as follows:

$$\mathbf{r}(k) = e^{\eta k} R \cos(\Theta + \nu)k \acute{\mathbf{x}}_1 - e^{\eta k} R \sin(\Theta + \nu)k \acute{\mathbf{x}}_2 + Ce^{-2\eta k} \acute{\mathbf{x}}_3$$

Take  $\eta > 0$ .

i. Let  $k \rightarrow -\infty$ .

$$\mathbf{r}(k) = Ce^{-2\eta k} \acute{\mathbf{x}}_3$$

It follows that  $\mathbf{x}_3$  defines the path of the spine.

ii. Let  $k \rightarrow \infty$ .

$$\mathbf{r}(k) \rightarrow Re^{\eta k} \cos(\Theta + \nu)k \acute{\mathbf{x}}_1 - Re^{\eta k} \sin(\Theta + \nu)k \acute{\mathbf{x}}_2$$

It follows that  $\mathbf{x}_1, \mathbf{x}_2$  defines the fan plane, and the field lines in the fan are spirals.

## 2.2 M is not diagonalisable

In this case, two of the eigenvalues are repeated and the matrix  $\mathbf{M}$  can only be reduced to the Jordan normal form,

$$\mathbf{J}_n = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}$$

Recalling that,

$$\frac{d\mathbf{r}(k)}{dk} = \mathbf{M} \cdot \mathbf{r}(k) = \mathbf{B} \quad (10)$$

we can again write  $\mathbf{r}(k) = \mathbf{P}\mathbf{u}(k)$ , where this time  $\mathbf{P} = (\mathbf{x}_1, \mathbf{x}_2^*, \mathbf{x}_3)$ , and  $\mathbf{x}_1, \mathbf{x}_2^*, \mathbf{x}_3$  satisfy

$$\begin{aligned} \mathbf{M}\mathbf{x}_1 &= \lambda\mathbf{x}_1 \\ \mathbf{M}\mathbf{x}_2^* &= \mathbf{x}_1 + \lambda\mathbf{x}_2^* \\ \mathbf{M}\mathbf{x}_3 &= -2\lambda\mathbf{x}_3 \end{aligned}$$

Then,

$$\frac{d\mathbf{u}(k)}{dk} = \mathbf{P}^{-1}\mathbf{M} \cdot \mathbf{P}\mathbf{u} = \mathbf{J}_n \mathbf{u} \quad (11)$$

And the equation for a field line is therefore:

$$\mathbf{r}(k) = (A + Bk)e^{\lambda k}\mathbf{x}_1 + Be^{\lambda k}\mathbf{x}_2^* + Ce^{-2\lambda k}\mathbf{x}_3 \quad (12)$$

where  $A, B, C$  are constant along a field line. Here, assume  $\lambda > 0$ .

i. Let  $k \rightarrow -\infty$ .

$$\mathbf{r}(k) = Ce^{-2\lambda k}\mathbf{x}_3$$

It follows that  $\mathbf{x}_3$  defines the path of the spine.

ii. Let  $k \rightarrow \infty$ .

$$\mathbf{r}(k) \rightarrow (A + Bk)e^{\lambda k}\mathbf{x}_1 + Be^{\lambda k}\mathbf{x}_2^*$$

It follows that  $\mathbf{x}_1$  and  $\mathbf{x}_2^*$  define the fan plane.

It is not difficult to see the reversal entailed by making  $\lambda < 0$ .

## 3 Reflections on the null's positive or negative character

It is apparent from what has been discussed above that, in the general case, the spine is seen to lie along the eigenvector of  $\mathbf{M}$  that related to the single eigenvalue of opposite sign to the real parts of the remaining eigenvalues. The remaining eigenvectors associated with these eigenvalues define the plane of the fan.

It is also apparent, with little further discussion, that if

i.  $Real(\lambda_1, \lambda_2) > 0$  then we have a *positive* neutral point. The field lines in the fan are directed away from the null, whereas the spine points into the null.

ii.  $Real(\lambda_1, \lambda_2) > 0$  then we have a *negative* neutral point. The field lines in the fan point into the null, whereas the spine is directed away from it.